

# *Implicit Submanifolds*

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Let  $M^n$  be a topological submanifold of  $R^{n+k}$ . Through the years, authors have introduced a variety of ‘niceness’ conditions on the way that  $M^n$  has been embedded, so that they can prove some reasonably strong results about  $M^n$ , *e.g.*  $M^n$  is a combinatorial submanifold,  $M^n$  has a transverse field,  $M^n$  is locally flat, etc. We are going to introduce an analytic condition on  $M^n$  which has some pleasing properties: we require that  $M^n$  be embedded so that the charts of  $M^n$  can be defined by implicit relations. We obtain the following insight into the class of submanifolds which can be smoothed locally by the Cairns–Whitehead technique ([2] and [3]):  $M^n$  has a transverse  $k$ -plane at each point if and only if the charts of  $M^n$  can be defined by Lipschitz implicit relations.

We next turn to the case where the manifold has a global implicit description. We show that such a manifold has a product normal microbundle neighborhood defined by a field of complementary  $k$ -planes, which may not be transverse, but which do have simple intersection with the manifold at the base point (we shall say that such complementary planes are *topologically transverse*). It then follows from the methods of [5; §4] that the manifold can be smoothed by arbitrarily small isotopic deformations. This theorem was proved by Whitney [3] under the additional hypothesis that the defining function is Lipschitzian. We conclude the paper with a discussion of submanifolds with codimension one which can be described locally by implicit relations where there is no transverse differentiability, only transverse monotonicity.

**1. The implicit function theorem.** The implicit function theorem was first proved by Cauchy for analytic functions. In this case, the regular level surfaces of the defining function are analytic submanifolds. Dini then showed that the theorem holds for  $C^r$  functions ( $r > 0$ ) and that in this case, the regular level surfaces are  $C^r$  submanifolds. These results are treated in standard advanced calculus texts. Young [6] then investigated some cases where the defining functions are not even  $C^1$ , and obtained some sufficient conditions for the level surfaces to be topological manifolds. We shall state two such theorems here. A

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