

*Lipschitz Spaces and Bernstein's Theorem on Absolutely Convergent Fourier Transforms**

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0. Introduction. The two most important results involving Lipschitz spaces are Bernstein's Theorem and Sobolev's Lemma. They are statements of the following form.

Bernstein's Theorem. $\mathcal{F} : \Lambda \rightarrow L_p$ (\mathcal{F} signifies Fourier transform);

Sobolev's Lemma. $\Lambda \subset L_a$, where Λ is an appropriate Lipschitz space.

There are various notions of "Lipschitz space" which have been considered. We shall introduce spaces $\Lambda_{a,p}^\alpha$ which are the simplest to define and yield the most powerful theorems. Our spaces $\Lambda_{a,p}^\alpha$ are closely related to the spaces $\Lambda(\alpha; a, p)$ studied by Taibleson [6]. When $\alpha > 0$, $\Lambda(\alpha; a, p) = \Lambda_{a,p}^\alpha \cap L_a$; the full details will be found in Appendix 2.

The precise definition of $\Lambda_{a,p}^\alpha$ is given at the beginning of Section 1. There are, up to equivalence, many ways of defining the $\Lambda_{a,p}^\alpha$ -norm. The easiest approach is to use a concrete definition in terms of differences. Given a function f , a positive integer k , and $1 \leq a \leq \infty$ we put $\omega(h) = \|\Delta^k(h)f\|_a$ where $\Delta^k(h)$ is the k th difference operator. For $0 < \alpha < k$ we define $\Lambda_{a,p}^\alpha(f)$ to be the \mathcal{L}_p -norm (using the dilation-invariant measure on $R^n - \{0\}$ in h of $|h|^{-\alpha} \omega(h)$). Note that the $\Lambda_{a,p}^\alpha$ -norm depends only on (arbitrarily high order) differences of the function. The $\Lambda_{a,p}^\alpha$ spaces with $\alpha > 0$ are the Banach spaces obtained by completing \mathcal{D} (infinitely differentiable functions of compact support) for the $\Lambda_{a,p}^\alpha$ -norm. The basic facts about the $\Lambda_{a,p}^\alpha$ spaces for $\alpha > 0$ are given in §1. It is technically convenient to extend the definition to $\alpha \leq 0$; this is done in §6 where we show that $f \in \Lambda_{a,p}^\alpha$ iff its derivatives of order β are in $\Lambda_{a,p}^{\alpha-\beta}$.

The multiplicity of indices may seem to be an annoyance, but they are really held to a minimum necessary for precise statements. Indeed, for Sobolev's Lemma and Bernstein's Theorem it is desirable to complicate the Lebesgue spaces. Sobolev's Lemma in the form $\Lambda \subset L_a$ (which includes the original

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