

# Joint Distributions of Observables

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**I. Introduction.** If  $f$  and  $g$  are random variables on a probability space  $(\Omega, \mathfrak{F}, p)$ , the joint distribution of  $f$  and  $g$  is defined as the Borel measure  $p_{f,g}$  on  $R^2$  where  $p_{f,g}(E) = p\{\omega: (f(\omega), g(\omega)) \in E\}$ . Similar definitions are given for joint distributions of any finite number of random variables and finite dimensional joint distributions may be defined for an arbitrary collection of random variables. If  $E, F$  are one-dimensional Borel sets then  $p_{f,g}(E \times F) = p(f^{-1}(E) \cap g^{-1}(F))$  may be thought of as the probability that  $f$  has a value in  $E$  and  $g$  has a value in  $F$ . Alternatively one may say that  $f$  and  $g$  have a joint distribution if there exists a measure  $\mu$  on the Borel sets  $B(R^2)$  such that  $\mu(E \times F) = p(f^{-1}(E) \cap g^{-1}(F))$  for every  $E, F \in B(R)$ . Using this definition it is obvious that a joint distribution  $\mu$  always exists, is unique, and  $\mu \equiv p_{f,g}$ . The situation is quite different when one considers observables in quantum mechanics. First one must decide how to define a joint distribution of observables. One soon finds that with any reasonable definition joint distributions do not always exist for observables. Thus a given collection  $C$  of observables can be classified into one of three different categories. If  $C$  always has finite joint distributions then we call  $C$  probabilistically definite, if  $C$  has no finite joint distributions we call  $C$  probabilistically indefinite, and otherwise we call  $C$  probabilistically mixed. In this paper two different types of joint distributions are considered and some of their properties are derived.

**2. Joint observables.** In this paper we shall assume that all observables are defined on a sum logic  $L$  (see [4] for definitions and notation.) If  $(\Omega, \mathfrak{F})$  is a measurable space, a  $\sigma$ -homomorphism  $h$  from  $\Omega$  into  $L$  is a map which satisfies

- (i)  $h(\Omega) = 1$ ; (ii)  $h(E) \perp h(F)$  if  $E \cap F = \emptyset, E, F \in \mathfrak{F}$ ;
- (iii)  $h(\cup E_i) = \sum h(E_i), E_i \cap E_j = \emptyset, i \neq j, E_i \in \mathfrak{F}$ .

Let  $x$  and  $y$  be observables on  $L$ . If there is a  $\sigma$ -homomorphism  $z$  from  $B(R^2)$  into  $L$  which satisfies  $z(E \times R) = x(E), z(R \times F) = y(F)$  for all  $E, F \in B(R)$ , then  $z$  is called a *joint observable* for  $x$  and  $y$ . We define joint observables for an arbitrary family of observables in an analogous fashion. To simplify matters in the