

Interpolation by Ritz Approximation

BERNIE L. HULME

Communicated by GARRETT BIRKHOFF

1. Introduction. We obtain below interpolation properties and sharpened error bounds for approximate piecewise polynomial solutions to

$$\begin{aligned} (1) \quad & u^{(2n)}(x) = r(x), \quad a \leq x \leq b, \quad n \geq 1, \\ (2) \quad & u^{(j)}(a) = 0, \quad u^{(j)}(b) = 0, \quad j = 0, \dots, n-1, \end{aligned}$$

obtained by variational methods. It is classic that the exact solution of (1) – (2), $\bar{u}(x)$, minimizes

$$(3) \quad J[u] = \frac{1}{2} \int_a^b [u^{(n)}(x)]^2 dx - (-1)^n \int_a^b r(x)u(x) dx.$$

over $K_0^{n,2}[a, b]$, the linear space of all real-valued functions $u(x)$ defined on $[a, b]$, such that $u^{(n-1)}(x)$ is absolutely continuous on $[a, b]$, $u^{(n)}(x) \in L^2[a, b]$, and $u(x)$ satisfies the homogeneous conditions (2).

Let $\pi: a = x_0 < x_1 < \dots < x_N = b$ be any partition of $[a, b]$ with $\bar{h} = \max \Delta x_i$. For fixed n and π consider a two-parameter family of spaces $F(\pi, p, 2m)$ consisting of functions $y \in C^{p-1}[a, b]$ which are polynomials of degree $2m - 1$ on each subinterval $[x_i, x_{i+1}]$, where $m \geq n$ and $n \leq p \leq 2m - 1$. Finally let $F_0(\pi, p, 2m) \subset F(\pi, p, 2m)$ denote the subspaces of functions which satisfy (2). We shall minimize J over each $F_0(\pi, p, 2m)$.

Using the notation

$$y^{(-j)}(x) \equiv \int_a^x \int_a^{t_1} \dots \int_a^{t_{j-1}} y(t_j) dt_j \dots dt_1, \quad j \geq 1,$$

we state in the following theorem our main result, which is an extension of the spline interpolation results of de Boor and Lynch [1].

Theorem 1. *The minimum of J on $F_0(\pi, p, 2m)$, where $m = n + q$, $q \geq 0$, and $n \leq p \leq 2m - 1$, is given by $\bar{y} \in F_0(\pi, p, 2m)$ which interpolates $\bar{u}^{(j)}(x_i)$ for all $x_i \in \pi$ and $j = -2q, \dots, 2n - p - 1$, as well as $\bar{u}^{(j)}(a)$ and $\bar{u}^{(j)}(b)$ for $j = -2q, \dots, n - 1$.*

2. Proof of Theorem 1 and error bounds. The proof of Theorem 1 requires