

# *Lipschitz Spaces and the Heat Equation*

B. FRANK JONES, JR.\*

*Communicated by* JURGEN MOSER

**1. Introduction.** Several authors have now treated a wide variety of classes of function spaces. For example, the survey article [10] contains some very general results on Lipschitz spaces of functions of  $n$  variables with smoothness conditions which can be different in each of the  $n$  coordinate directions. In case the smoothness conditions are rotationally invariant, there are intimate connections between the function spaces which arise and the properties of solutions of Laplace's equation in the half space  $\mathbf{R}^n \times (0, \infty)$  with boundary values in the given function spaces. See [15], [16], [17] for a detailed exposition of these properties. The purpose of this paper is to obtain the similar development in case the function spaces have one coordinate treated differently than the others, and to relate certain of these spaces to solutions of the heat equation in a half space. In anticipation of this paper some results in this direction have already been given [9], which mainly involve the theory of singular integral operators with mixed homogeneity [1], [4], [8].

In fact, the key to almost everything in this paper is the mixed homogeneity. The fact that solutions of the heat equation  $u_t = u_{x_1x_1} + \dots + u_{x_nx_n} + u_{yy}$  are invariant under the transformations  $x \rightarrow \lambda x$ ,  $y \rightarrow \lambda y$ ,  $t \rightarrow \lambda^2 t$ , indicates that changes of variables of this form should be used in the analysis. Therefore, expressions for norms etc. should be so constructed as to be invariant under such transformations. Thus, for example, the measure  $(|x| + |t|^{1/2})^{-n-2} dx dt$  appears frequently.

Since the motivation for this paper arises from the desire to correlate smoothness properties of functions which are to act as boundary values for solutions of the heat equation with the behavior of the solutions near the boundary, we have taken the somewhat unusual approach of considering the *pure* boundary value problem—*i.e.*, no initial condition is to be assigned. Thus, the solutions of the heat equation in this paper will always exist for  $-\infty < t < \infty$ , rather than the usual  $0 < t < \infty$ . It is trivial to observe that if the boundary values happen to vanish for  $t < 0$ , then the solution can be viewed as a solution of the mixed problem with boundary values specified for  $t > 0$  and initial values (at  $t = 0$ ) identically zero.

---

\* Research sponsored by the Air Force Office of Scientific Research, Office of Aerospace Research, United States Air Force, under AFOSR Grant No. 941-65.