

# On Hölder Continuity at the Boundary in Conformal Maps\*

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1. While there are various criteria for differentiability of the Riemann mapping function at the boundary of its domain of definition, it is useful to have a condition which requires less than these and yields the Hölder continuity of this function. Such a condition is presented here and moreover the Hölder exponent obtained is the best possible under the hypotheses.

Let  $C$  be a closed rectifiable Jordan curve with bounded arc length-chord length ratio, *i.e.* for any two points  $z_1, z_2 \in C$ , the length  $\Delta s$  of the shorter arc of  $C$  between them satisfies  $\Delta s \leq c |z_1 - z_2|$ , where  $c > 1$  is a constant. If  $\varphi$  maps the interior of  $C$  conformally onto the unit disk  $|w| < 1$ , we show that  $\varphi \in \text{Lip}(\mu)$  on  $C$  where  $\mu = \frac{1}{2}(1 - (1/\pi) \arcsin 1/c)^{-1}$  and that this exponent is sharp.

Domains with a boundary curve of bounded arc length-chord length ratio play a useful role in various applications of conformal mapping. If  $f$  denotes the *inverse* function of  $\varphi$ , this property, for example, ensures that  $f$  satisfies the Smirnov condition, *i.e.* that  $\log |f'(w)|$  is represented by its Poisson integral in  $|w| < 1$  [5, p. 183]; this condition is important in the theory of Szegő polynomials connected with arbitrary rectifiable Jordan curves. Furthermore,  $f(w)$  also satisfies a Hölder condition on  $|w| = 1$ , where the exponent depends on  $c$  [8]. Other properties which this mapping possesses have been given in [4] (see Remark 2 below) and [9].

**2. Theorem.** *Suppose  $C$  is a closed rectifiable Jordan curve in the  $z$ -plane such that*

(a) *for any two points  $z_1, z_2 \in C$  and the length  $\Delta s$  of the (shorter) arc of  $C$  between them*

$$(1) \quad \Delta s \leq c |z_2 - z_1|, \quad c > 1, \quad c = \text{constant}$$

(b) *the interior  $\Omega$  of  $C$  contains the disk  $|z| \leq 2\sigma, \sigma > 0$ .*

*Let  $w = \varphi(z)$  map  $\Omega$  conformally onto the unit disk such that  $\varphi(0) = 0$ . Let  $\rho_0 =$*

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