

On the Pointwise Behavior of Iterates of a Self-adjoint Operator

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The question with which this paper is concerned is the following: Let T be a self-adjoint, linear operator on L_1 of a measure space of finite measure. Assume that $\|T\|_1 \leq 1$. Then if f is in L_1 does $T^{2^n}f$ converge pointwise a.e.?

A case of special interest arises in the following way: Let T_1 and T_2 be operators defined by taking the conditional expectation with respect to σ fields θ_1 and θ_2 . (Then $T_1 T_2 T_1 = T$ is selfadjoint and $T^n = (T_1 T_2)^n T_1$.) One would then expect $(T_1 T_2)^n T_1 f$ and $(T_2 T_1)^n f$ to converge pointwise a.e. (and that the limit would equal the conditional expectation with respect to $\theta_1 \wedge \theta_2$).

Burkholder and Chow [1] showed that the answer to these questions is yes if f is in L_2 . Rota [2] and E. M. Stein [3] showed the result to be true for functions in L_p , $p > 1$. (Rota's method gives something much more general.) In this paper we will show that the answer is no, even in the special case mentioned above, if f is only assumed to be in L_1 .

Theorem. Let I be the unit interval. Then there are 2 σ -fields, θ_1 and θ_2 consisting of Borel sets in I , and a function f , defined on I , $f \geq 0$, $\int f \leq \frac{1}{8}$. If T_1 and T_2 are the operators: conditional expectation with respect to θ_1 and θ_2 respectively, then $\lim_{n \rightarrow \infty} (T_1 T_2)^n f \geq 1$ a.e. on I .

Remark 1. The above theorem implies that $(T_1 T_2)^n f$ does not converge pointwise a.e..

Remark 2. It will follow easily from the construction that $(T_1 T_2 T_1)^{2^n} g$ does not converge pointwise a.e. for some g in L_1 .

Definition. By an (X, r, s) system we will mean the following: X is a measure space of finite measure, r a finite collection of disjoint subsets of X whose union is X , s a finite collection of disjoint subsets of X (whose union is not X).

r will give rise to an operator, R ; conditional expectation with respect to r . s will give rise to an operator, S as follows; S will be the identity on functions whose support is disjoint from $\cup s$ and for functions with support in $\cup s$, S will be conditional expectation with respect to s .

We will say that (X, r, s) is ergodic if for any function g , $(RS)^n g$ tends to a constant.