

# *Existence and Uniqueness of Solutions to a Nonlinear Boundary Value Problem*<sup>1</sup>

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Often in questions of the existence and uniqueness of solutions of the two point boundary value problem

$$(1) \quad y'' + f(t, y, y') = 0$$

$$(2) \quad y(t_1) = r_1$$

$$(3) \quad y(t_2) = r_2$$

the conditions on  $f(t, y, y')$  are such that uniqueness of solutions, *i.e.* existence of at most one solution, is easy to establish but existence of at least one solution is more difficult, see [1]–[2], for example. Recently Lasota and Opial [3] showed that under the assumption

- (c) For each point  $(t_0, x_0, u_0) \in D$ , the domain of definition of  $f$ , there exists one and only solution  $x(t) = x(t; t_0, x_0, u_0)$  of equation (1) defined on  $(a, b)$  such that  $x(t_0) = x_0, x'(t_0) = u_0,$

that the existence of at most one solution of the boundary value problem (1) (2) (3) for every pair  $(t_1, r_1)(t_2, r_2)$  of  $(a, b) \times R$  implies that there exists at least one solution of (1) (2) (3). Here we remove the uniqueness requirement for solutions of initial value problems but assume that all solutions are continuable over  $(a, b)$ . This is important because it removes the need for any sort of Lipschitz type restriction on the nonlinearity.

**Theorem.** *Suppose  $f(t, y, y')$  is continuous on  $(a, b) \times R \times R$  and that solutions of (1) with initial conditions at  $t_0 \in (a, b)$  can be continued over  $(a, b)$ . If there exists at most one solution to the boundary value problem (1) (2) (3) for every choice  $(t_i, r_i) \in (a, b) \times R, i = 1, 2$ , there exists exactly one solution of (1) (2) (3).*

**Remark.** Actually, the proof can be made to work on  $[a, b]$  or on  $(a, b]$ .

**Proof.** Let  $y(t, r, u)$  be a solution of equation (1) which satisfies  $y(t_1) = r_1, y'(t_1) = u$ . Define a point to set mapping  $\Gamma$  from  $R$  into  $R$  by associating with

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