

Discussion and Extension of a Theorem of Tricomi Concerning Functions which Assume All Intermediate Values

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1. Introduction. The intermediate value property of continuous real valued functions of a real variable is, as is well known, not characteristic of continuous functions. That is to say, there are known examples of discontinuous functions which possess this property. In seeking to obtain what he called "better examples", Tricomi [1] was led to the discovery of an interesting elementary theorem, which states that any *bounded* function which has this property, plus just a little bit more (that is, "plus an epsilon more", so to speak), must necessarily be continuous and monotone (for a quotation of Tricomi's own formulation of his theorem, see below, under subsection T1 of Section 2).

The present paper contains an extension of Tricomi's theorem, since here the hypothesis that the functions in question be bounded in absolute value is not made, see Theorems 1 and 2 of Section 3. The intermediate Section 2 contains background and methodological remarks concerning functions which possess the intermediate value property. These remarks are grouped under two subheadings: H1 to H4 are relative to G. H. Hardy's book [2], while T1 to T5 are relative to Tricomi's paper [1]. While these remarks are of specific logical and mathematical interest, a detailed examination of Section 2 is not needed for the understanding of Section 3, and the reader may wish to pass directly from this introduction to Section 3.

2. Discussion of Tricomi's theorem.

H1. In section 101, page 190, of [2], entitled "The fundamental property of a continuous function", G. H. Hardy states: "... a continuous function $f(x)$ should possess the following property: if

$$f(x_0) = y_0, f(x_1) = y_1,$$

and $y_0 < y < y_1$, then there is a value of x between x_0 and x_1 for which $f(x) = y$. In other words, as x varies from x_0 to x_1 , f must assume at least once every value between y_0 and y_1 ."