

# Quasiconformal Mappings of Slit Domains in Three Space

F. W. GEHRING<sup>(1)</sup>

**1. Introduction.** This paper is concerned with the following basic problem. Characterize those domains  $D$  in the  $n$ -dimensional Möbius space  $\bar{R}^n$  which can be mapped quasiconformally onto the open unit ball  $B^n$ . In [5] it was shown that to solve this problem one need only look at the part of  $D$  near its boundary  $\partial D$ . That is, a domain  $D \subset \bar{R}^n$  is quasiconformally equivalent to  $B^n$  if and only if  $\partial D$  has a neighborhood  $U$  such that  $D \cap U$  can be mapped quasiconformally into  $B^n$  with  $\partial D$  corresponding to  $\partial B^n$ .

When  $n = 2$ , the Riemann mapping theorem leads to a geometric characterization which is more satisfactory than the condition implicit in the above mentioned result. Namely, a domain  $D \subset \bar{R}^2$  can be mapped quasiconformally onto  $B^2$  if and only if  $\partial D$  is a continuum which contains at least two points. Unfortunately when  $n \geq 3$ , one cannot decide whether or not a domain  $D \subset \bar{R}^n$  is quasiconformally equivalent to  $B^n$  by looking only at  $\partial D$ . For example, let

$$D_1 = \{x : x_n > \min(r^{1/2}, 1)\}, \quad D_2 = \{x : x_n < \min(r^{1/2}, 1)\},$$

where  $r = (x_1^2 + \cdots + x_{n-1}^2)^{1/2}$ . Then  $D_1$  and  $D_2$  are Jordan domains in  $\bar{R}^n$  with  $\partial D_1 = \partial D_2$ , a trivial modification of the proof of Theorem 10.3 in [6] yields a quasiconformal mapping of  $D_1$  onto  $B^n$ , while the  $n$ -dimensional analogue of Theorem 10.1 in [6] shows that  $D_2$  is not quasiconformally equivalent to  $B^n$ .

The trouble in the above example arises because the common boundary is quasiconformally good on one side but not on the other. This suggests that when  $n \geq 3$ , one should begin by considering domains  $D \subset \bar{R}^n$  whose boundaries have only one side. In the present paper, we study those domains  $D \subset \bar{R}^3$  whose complements lie in a set  $S$ , homeomorphic to  $\partial B^3$ . This class, while extremely special, is a natural analogue in  $\bar{R}^3$  of the important class of simply connected slit domains in  $\bar{R}^2$ . We prove in Section 7 that each such domain  $D$  is quasiconformally equivalent to  $B^3$  if and only if  $\partial D$  is a quasiconformal disk, that is, the image of the closed unit disk  $\bar{B}^2$  under a quasiconformal mapping of  $\bar{R}^3$  onto

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<sup>(1)</sup> This research was supported in part by the National Science Foundation, Contract GP-7234.