

# *A Perturbation Theorem for Invariant Manifolds and Hölder Continuity\**

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**1. Introduction.** We are concerned with a compact Riemannian manifold  $M_0$  without boundary imbedded in a Riemannian manifold  $S$ , and a vector field  $V_0$  defined in an  $S$  neighborhood of  $M_0$ . We assume  $M_0$  is an invariant manifold (integral manifold) for  $V_0$ , i.e.,  $V_0$  is tangential to  $M_0$  at each point of  $M_0$ . The problem is to find conditions under which the slightly perturbed vector field  $V$  will have an invariant manifold  $M$  near  $M_0$ , and to predict the smoothness of  $M$ .

It is well known that it does not suffice to merely require  $V$  to be close to  $V_0$ . One must impose additional conditions on the vector field  $V_0$  near  $M_0$ . We give such conditions in terms of two invariantly defined functions  $\lambda(x)$  and  $\bar{\beta}(x)$  defined on  $M_0$  which are related to the type numbers of Lyapunov and are determined by the vector field  $V_0$  solely in the neighborhood of the limit sets of the flow on  $M_0$ .  $\lambda$  measures the rate of attraction of solutions in  $M_0$  toward limit sets and  $\bar{\beta}$  measures the rate of attraction toward  $M_0$  of nearby solutions in the neighborhood of limit sets on  $M_0$ . It will be shown that if (1)  $\bar{\beta}(x) < 0$  and (2)  $\bar{\beta}(x) < (q + \alpha)\lambda(x)$ ,  $q \geq 1$  an integer,  $0 < \alpha < 1$ , then under a sufficiently small perturbation a slightly perturbed vector field  $V$  will possess an invariant manifold  $M$  which is a  $q$ -times differentiable manifold and the  $q^{\text{th}}$  derivative satisfies a Hölder condition with exponent  $\alpha$ . The greater the attraction toward  $M_0$  the more negative  $\bar{\beta}$  will be and the greater the attraction of solutions in  $M_0$  toward limit sets, the more negative  $\lambda$  will be. Thus any lack of smoothness is due to the presence of attracting limit sets on  $M_0$ .

In cases in which the limit sets are simple, e.g., singularities or limit cycles, one can check conditions (1) and (2) by comparing eigenvalues and Floquet exponents of the linearized equations (Example 7-4).

If  $\bar{\beta}(x) = \lambda(x)$  for some  $x = x_0$ , then under a perturbation the manifold may roll up into a spiral at  $x_0$  (Example 7-3). If  $\bar{\beta}(x) < \lambda(x)$  then by semicontinuity

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