

Some Existence Theorems for Quasi-linear, Non-uniformly Elliptic Equations in Divergence Form

NEIL S. TRUDINGER*

Communicated by D. GILBARG

1. Introduction. Let Ω be a bounded domain in E^n . The spaces $C^{k,\alpha}(\Omega)$ $\{C^{k,\alpha}(\bar{\Omega})\}$, for k a non-negative integer and $0 < \alpha < 1$, consist of functions k times continuously differentiable in $\Omega\{\bar{\Omega}\}$ having their derivatives of order k locally Hölder continuous, with exponent α , in $\Omega\{\bar{\Omega}\}$. We write $C^{k,0}(\Omega) = C^k(\Omega)$ and $C^{0,\alpha}(\Omega) = C^\alpha(\Omega)$. The operator

$$(1) \quad Qu = a^{ij}(x, u, u_x)u_{x_i x_j} + a(x, u, p)$$

acting on functions $u(x) \in C^2(\Omega)$ is *elliptic* in Ω if the minimum eigenvalue $\lambda(x, u, p)$ of the matrix $A(x, u, p) = [a^{ij}(x, u, p)]$ is positive in $\Omega \times E \times E^n$. Here

$$u = (u_{x_1} \cdots u_{x_n}), \quad p = (p_1, \cdots p_n)$$

and repeated indices indicate summation from 1 to n . The functions $a^{ij}(x, u, p)$ are defined in $\Omega \times E \times E^n$. If furthermore for any $M > 0$, the ratio of the maximum to minimum eigenvalues of $A(x, u, p)$ is bounded in $\Omega \times (-M, M) \times E^n$, then Qu is called *uniformly elliptic*. When Qu is elliptic but not necessarily uniformly elliptic, it is referred to as *non-uniformly elliptic*. The operator Qu is of *divergence form* if there exists a vector function $\hat{a}(x, u, p) = \hat{a}^i(x, u, p)$ such that $a^{ij}(x, u, p) = \partial/\partial p_i \{ \hat{a}^i(x, u, p) \}$, i.e.

$$(2) \quad Qu = \text{div. } \hat{a}(x, u, u_x) + a(x, u, u_x)$$

where $\text{div} = \Sigma^n \partial/\partial x_i$.

Our concern here is with non-uniformly elliptic divergence structure equations. In fact, no restrictions on the maximum eigenvalues of $A(x, u, p)$ will be present in our theorems. The problem is to determine growth conditions satisfied by the inhomogeneous terms of (2) with respect to $\lambda(x, u, p)$ so that the Dirichlet problem for Qu and suitable boundary conditions is solvable; a solution of the

* This research was partially supported by Air Force Contract AF 49 (638) 1345 at Stanford University during the academic year 1965–66.