

Uniqueness Theorems for Axially Symmetric Partial Differential Equations

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I. Introduction. Although existence and uniqueness theorems for linear elliptic partial differential equations in a domain D with coefficients continuous in D have been known for some time [2], similar results for equations whose coefficients have singularities in the domain under consideration are practically unknown. Recently attention has been given to a class of singular equations which appear frequently in both pure and applied mathematics and are known as generalized axially symmetric partial differential equations. Typical examples of equations from this class are the following:

$$\begin{aligned} (1) \quad & \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{2\nu}{y} \frac{\partial u}{\partial y} = 0, \\ (2) \quad & \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{2\nu}{y} \frac{\partial u}{\partial y} + k^2 u = 0, \\ (3) \quad & \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{2\nu}{y} \frac{\partial u}{\partial y} - k^2 u = 0. \end{aligned}$$

Here ν and k are real numbers, $k > 0$. Just as a thorough knowledge of Laplace's equation guided the attack on linear elliptic equations with continuous coefficients, it is hoped that a better understanding of equations such as 1), 2) and 3) will give insight towards developing a theory of elliptic equations with singular coefficients.

The results so far have been sparse and incomplete. In 1965 Parter [18] derived existence and uniqueness theorems for equation 1) (known as the generalized axially symmetric potential or GASP equation) for the case in which $\nu \geq -\frac{1}{2}$ and showed that his results were no longer valid for $\nu < -\frac{1}{2}$. In 1968 Colton [1] proved a uniqueness theorem for the exterior Dirichlet problem for equation 2) (the generalized axially symmetric Helmholtz or GASH equation) for $\nu > -\frac{1}{2}$.

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