

# *The General Similarity Solution of the Heat Equation*

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**1. Introduction.** A systematic approach is given for finding similarity solutions to partial differential equations and, in particular, the heat equation, by the use of transformation groups. New solutions to the heat equation are obtained.

The application of group theory to the solution of partial differential equations was first considered by Lie [1] and later by Ovsjannikov [2] and Matschat and Müller [3]. If a one-parameter group of transformations leaves invariant an equation and its accompanying boundary conditions, then the number of variables can be reduced by one. The functional form of the solution can be deduced by solving a first order partial differential equation derived from the infinitesimal version of the global group. The functional form of the solution for two independent variables  $(x, t)$ , say, is

$$(1) \quad u(x, t) = F(x, t, \eta, f(\eta))$$

where  $\eta$  is called the similarity variable. The dependence of  $F$  on  $x, t, \eta(x, t)$  (for calculation purposes, as will be seen later, it is convenient to isolate  $\eta$ ), and  $f(\eta)$  is known explicitly from invariance considerations.  $f(\eta)$  satisfies some ordinary differential equation obtained by substituting the form (1) into the given partial differential equation. The form (1) is called the general similarity solution. Initially no special boundary conditions are imposed since eventually we use the invariants of the group to establish the boundary conditions.

**2. Formulation of invariance.** Let

$$(2) \quad u = \theta(x, t)$$

be a solution of a certain partial differential equation

$$(3) \quad \mathfrak{M}[u] = 0$$

defined over a region  $R$  in the  $(x, t)$  plane (Fig. 1) on which boundary conditions