

On the Conformal Equivalence of Compact 2-Dimensional Manifolds

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In this note we obtain a new proof of the fact that any orientable compact non-simply connected 2-manifold with a given smooth (C^∞) Riemannian metric is conformally equivalent to a manifold of constant nonpositive curvature. The proof proceeds by means of the theory of nonlinear elliptic partial differential equations and the calculus of variations. It is interesting to note that the proofs given here are in the spirit of the famous paper of Poincaré [1]. At the end of the paper we show how to modify our proof in the simply connected case to prove conformal equivalence to a sphere. Since the isometry classes of 2-manifolds with constant curvature are well-known, we obtain a new proof of the uniformization theorem for closed Riemann surfaces.

1. Let \mathfrak{M} be a compact 2-dimensional orientable manifold with a given smooth (C^∞) Riemannian metric $ds^2 = g_{11} dx_1^2 + 2g_{12} dx_1 dx_2 + g_{22} dx_2^2$ and with Gaussian curvature $R(x)$. Then if \mathfrak{M} is non-simply connected, by the Gauss-Bonnet formula

$$(1) \quad \int_{\mathfrak{M}} R(x) dS \leq 0$$

with equality holding if and only if the genus of \mathfrak{M} is 1. Furthermore, if we set $\bar{g}_{ij} = e^{2\sigma} g_{ij}$ ($i, j = 1, 2$) (*i.e.* we consider a conformally equivalent metric \bar{g}_{ij} determined by the conformal mapping $e^{2\sigma}$) then we obtain the following equation for σ

$$(2) \quad \Delta\sigma - R(x) + R'e^{2\sigma} = 0$$

where Δ is the Laplace Beltrami operator associated with g_{ij} and R' is the Gaussian curvature of the manifold \mathfrak{M} with the Riemannian metric \bar{g}_{ij} . (Indeed

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