

# *A Generalization of the Morse Index Theorem to a Class of Degenerate Elliptic Operators*

L. M. SIBNER

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In a recent paper, J. J. Kohn and L. Nirenberg show that some of the fundamental properties of strongly elliptic operators also hold for a class of degenerate elliptic operators (see [1]). A simple example which is typical of some of those they consider is the second order operator in the plane having principal part

$$y^2 \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

which is elliptic except on the real axis.

The purpose of this paper is to show that the classical Sturm–Liouville theory is valid for problems satisfying the Kohn–Nirenberg conditions. In particular, the proof of the Morse Index Theorem for strongly elliptic operators due to Smale (see [2]) can be modified for an appropriate class of degenerate problems.

We consider vector valued functions  $u = (u_1, \dots, u_n)$  defined in the closure of a bounded open set  $M$  in  $R^n$  having smooth boundary. Let  $B$  be the subspace of  $C^\infty(\bar{M})$  consisting of vector-valued functions which vanish up to order  $k - 1$  on  $\partial M$ . We will be concerned with linear differential operators  $L$  of order  $2k$  defined on  $B$ . For every  $u, v \in B$ , the  $L_2$  inner product  $(Lu, v)$  defines a bilinear form  $Q(u, v)$  in the derivatives of  $u$  and  $v$  up to order  $k$ ; i.e.,

$$(Lu, v) = Q(u, v) = \int_M \sum_{i,j} \sum_{\substack{|\alpha| \leq k \\ |\beta| \leq k}} a_{ij}^{\alpha\beta} D^\alpha u_i \overline{D^\beta v_j} dx$$

with smooth matrix valued coefficients.  $L$  is called *self-adjoint* if  $Q(u, v)$  is Hermitian.

The *Morse Index*  $I(L)$  is then defined as the dimension of the largest subspace of  $B$  on which the quadratic form  $Q(u, u)$  is Hermitian negative definite.

A *smooth deformation* of  $M$  is a family of domains  $M_t$ ,  $0 \leq t \leq 1$ , such that  $M_0 = M$  and  $M_t$  is properly contained in  $M$ , for  $t > 0$ . Moreover, the boundary of  $M_t$  depends on  $t$  in a smooth way. Such a deformation is of  $\epsilon$ -type if the measure of  $M_1$  is smaller than  $\epsilon$ . We will always assume that for every  $\epsilon > 0$ ,