

Distortion Theorems for the Level Curves of Rational Functions and Harmonic Functions

DOROTHY BROWNE SHAFFER

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I. Introduction. The level loci of Green's function and more general harmonic functions have certain properties of smoothness and limitations on distortion. These properties of the level curves and orthogonal trajectories of Green's function, including multiple points, tangents, normals and convexity have been studied by J. L. Walsh in a series of publications [4, 5, 6]. In a recent paper [3] the author studied the distortion and location of the circle of curvature of lemniscates and the level curves of Green's function. It is the object of this paper to extend these results to general rational functions and harmonic functions. The level loci of a harmonic function, are by no means arbitrary as to their shape, but also have certain limitations on distortion. The present investigation establishes best possible geometric bounds for these properties for rational functions, all of whose zeros and poles are contained in disjoint discs. The same bounds are shown to be valid for functions harmonic in a region bounded by sets of curves C_{1i} , $i = 1, \dots, m$ and C_{2i} , $i = 1, \dots, n$; $u(x, y) = 1$ on the set $C_{1i} \subset \text{disc } D_1$ and $u(x, y) = 0$ on $C_{2i} \subset \text{disc } D_2$.

II. Normals and bisectors. The foundation for the theory of the critical points of a rational function of a complex variable z :

$$(1) \quad R(z) = \frac{\prod_{i=1}^m (z - \alpha_i)}{\prod_{i=1}^n (z - \beta_i)}$$

was provided by Bôcher [1] who modified the field of force in Gauss' theorem by admitting both positive and negative masses. The lines of force are the loci $\arg R(z) = k$ which are orthogonal to the level curves R_μ :

$$(2) \quad |R(z)| = \frac{\prod_{i=1}^m |z - \alpha_i|}{\prod_{i=1}^n |z - \beta_i|} = \mu > 0.$$