

A Lemma of H. Tong and Some of Its Consequences

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We present here a proof of a lemma of H. Tong, which does not depend on the Axiom of Choice or any of its equivalents, and some well known results which are easy or immediate consequences of this lemma. Some of these results were announced in [1]. A much broader study was indicated in [1]. However, because of recent interest in this subject, it seems appropriate to publish some results at present, leaving further consequences and generalizations for another occasion.

Tong's Lemma. *Let g be a mapping from a subset of $P(X)$ (the set of all subsets of X) into X . Let $A \subset X$ be a well-ordered set with the property that $g(S_A(x)) = x$ for every $x \in A$ (where $S_A(x)$ is the (possibly empty) segment in A consisting of all the elements of A which are $< x$). Then the union U of all A satisfying the above conditions can be well-ordered in such a way that $g(S_U(x)) = x$ for every $x \in U$. Moreover, if $g(U)$ exists, then $g(U) \in U$.*

Remarks. We note that if $A_1 \neq A_2$ and both A 's satisfy the conditions on A given above, then one is a segment of the other (this will be shown in the proof below). It is also useful to note the conditions (on the well-ordered set U): (1) $g(S_U(x)) = x$ for every $x \in U$, and (2) If $g(U)$ exists, then $g(U) \in U$. The fact that these two conditions determine U uniquely can be readily seen.

Proof. (This proof and the proof of Corollary 1 are due to H. Tong.) Let A_1 and A_2 be sets satisfying the hypothesis. Let B be the set of all $x \in A_1 \cap A_2$ such that $S_{A_1}(x) = S_{A_2}(x)$. If $B \neq A_i$ for $i = 1, 2$, let a_i be the least element of $A_i - B$. Thus $S_{A_i}(a_i) = B$. Consequently, $g(B) = g(S_{A_i}(a_i)) = a_i$. Therefore, $a_1 = a_2$. This contradicts the definition of a_i . Thus, $B = A_1$ or A_2 . It follows that if $A_1 \neq A_2$, then one is a segment of the other.

We now define an ordering on U . Let x_1 and $x_2 \in U$. There exist A_i such that $x_i \in A_i$. The above result implies that one of the A 's must contain both x_1 and x_2 . Provide x_1 and x_2 with the ordering obtained in that A . As a consequence of the same result above, the ordering is independent of A , and it is easy to see that the U is thus well-ordered.