

# *A Completeness Theorem on the Group of Baire Equivalences*

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In a previous work [1], the first author considered the totality  $T$  of compact metric topologies  $\tau$  on a base set  $E$  which determine the same group  $\mathcal{G}$  of Baire equivalences. A Baire equivalence  $g$  is a bijective map which, along with its inverse map  $g^{-1}$ , maps Baire sets into Baire sets. If  $\tau$  is one of the topologies in  $T$ , it is shown how to introduce a uniform topology  ${}^{\tau}\mathcal{G}$  into the group of Baire equivalences. (Because of the group structure, there is also a right topology  $\mathcal{G}_{\tau}$ ; we shall concern ourselves only with the left topology). A principal result of [1] is that the uniform space  ${}^{\tau}\mathcal{G}$  is complete. The purpose of the present note is to introduce a weaker topology  ${}^{\tau}\mathcal{G}$  into the group of Baire equivalences and to establish once more the fact of completeness.

Let  $e$  denote the identity element in  $\mathcal{G}$ . Let  $\tau$  be a compact metrizable topology on  $E$ . Let  $f_1, \dots, f_n$  be  $\tau$ -continuous real-valued functions. For a fixed  $i$ ,  $1 \leq i \leq n$ , consider sets of the type  $M_i(\alpha) = \{x : f_i(x) = \alpha\}$ . We consider Baire equivalences  $h$  which permute these sets  $M_i(\alpha)$ . Thus, in particular, if  $h$  maps  $M_i(\alpha)$  into  $M_i(\beta)$  the two sets have the same cardinality. Now consider the totality of Baire equivalences  $h$  which permute the sets  $M_i(\alpha)$  for each  $i = 1, \dots, n$ . This totality is a subgroup  $\mathcal{S}$  of  $\mathcal{G}$  which will be denoted by  $\mathcal{S}(\tau; f_1, \dots, f_n)$ . We note that  $\mathcal{S}(\tau; f_1, \dots, f_n; f'_1, \dots, f'_m) = \mathcal{S}(\tau; f_1, \dots, f_n) \cap \mathcal{S}(\tau; f'_1, \dots, f'_m)$ . If we vary the set  $\{f_1, \dots, f_n\}$  in all possible ways we obtain a base of neighborhoods of  $e$  in  $\mathcal{G}$ . The topology  ${}^{\tau}\mathcal{G}$  is obtained by taking as a base of neighborhoods of  $g \in \mathcal{G}$  the sets  $g \cdot \mathcal{S}$  where  $\mathcal{S}$  is any set in the base for  $e$ . (Similarly, the sets  $\mathcal{S} \cdot g$  define the topology  $\mathcal{G}^{\tau}$ ).

The topology  ${}^{\tau}\mathcal{G}$  comes from a uniformity. The subsets  $\bigcup_i (g \cdot \mathcal{S} \times g \cdot \mathcal{S})$  of  $\mathcal{G} \times \mathcal{G}$  where  $g \in \mathcal{G}$  form a base of this uniformity. It is not difficult to see that the topology  ${}^{\tau}\mathcal{G}$  is separated. In fact, if  $g \neq e$  there exist  $x$  and  $y$  such that  $y = gx \neq x$ . Let  $f$  be  $\tau$ -continuous and such that  $f(x) = f(z)$  has a unique solution  $z = x$  whereas  $f(y) = f(z)$  has at least two solutions  $z$ . Then  $g \notin \mathcal{S}(\tau; f)$ . In order to establish this fact we have assumed that  $E$  possesses more than two points. We make this assumption in all that follows.

A generalized Cauchy sequence in  $\mathcal{G}$  is a set  $\{g_m\}$  indexed by a directed set  $M$  and having the property that given any neighborhood  $\mathcal{S} = \mathcal{S}(\tau; f_1, \dots, f_n)$

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