

# On the Inversion of Toeplitz Operators

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**1. Introduction.** Let  $m$  denote the normalized Haar measure on the circle group  $\mathbf{T}$ . For  $1 \leq p \leq \infty$ ,  $L^p$  will denote the usual (complex) Lebesgue space of this measure space. The closed subspace of  $L^p$  consisting of all functions whose Fourier coefficients vanish on the negative integers will be denoted by  $H^p$ . For  $g \in L^\infty$ , the *Toeplitz operator*  $T_g$  is an operator on  $H^2$  (bounded linear transformation of  $H^2$  into itself) defined by

$$T_g f = P(gf), \quad f \in H^2$$

where  $P$  is the orthogonal projection of  $L^2$  onto  $H^2$ . Widom ([15], [16]) obtained necessary and sufficient conditions on  $g$  in order that  $T_g$  is an invertible operator *i.e.* a bijective mapping. Independently, Devinatz obtained the inversion theorem in [1]. Devinatz shows that the theorem is valid for  $H^2$ -spaces of Dirichlet algebras. Their final result is the following:

*$T_g$  is invertible if and only if  $g$  can be written in the form  $\gamma \exp [u + i(v + \bar{w})]$ , where  $\gamma$  is a complex number of modulus 1,  $u, v, w$  are real functions in  $L^\infty$ ,  $\|v\|_\infty < \pi/2$  and  $\bar{w}$  denotes the conjugate function of  $w$ .*

In this paper we study the problem of invertibility of Toeplitz operators induced by essentially bounded operator valued functions on the  $H^2$  space of functions taking values in a complex separable Hilbert space. We shall show in §4 that an appropriate reformulation of the above result holds in the vectorial case. This reformulation states that  $T_g$  is invertible if and only if  $g$  can be factored in the form  $uh$  where  $h$  and  $h^{-1}$  are in  $H^\infty$ ,  $u$  is a function of modulus 1 almost everywhere and there exists an invertible element of  $H^\infty$  whose  $L^\infty$ -distance from  $u$  is less than 1. The equivalence of the two statements is a consequence of the formula for outer functions [8, p. 62] and a Lemma of Helson and Szegő [6, p. 126, Lemma 1].

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