

Positive Solutions of Some Nonlinear Eigenvalue Problems*

HERBERT B. KELLER

California Institute of Technology

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1. Introduction. We consider nonlinear eigenvalue problems of the form

$$(1.1a) \quad Lu + \lambda r(x)u = f(\lambda, x, u), \quad x \in D;$$

$$(1.1b) \quad Bu \equiv \beta_0(x)u + \beta_1(x) \frac{\partial u}{\partial \nu} = 0, \quad x \in \partial D;$$

where L is a second order self-adjoint elliptic operator. We seek positive solutions, $u(x) > 0$ on D , and investigate the set of values of λ for which such solutions exist. The nonlinearity is required to satisfy, for appropriate values of λ ,

$$f(\lambda, x, 0) \leq 0 \quad \text{on } D.$$

The case $f(\lambda, x, 0) \equiv 0$, which we include, has the trivial solution, $u(x) \equiv 0$, for all values of λ and thus leads to a bifurcation problem. Under various monotonicity conditions on $f(\lambda, x, z)$ and $f_z(\lambda, x, z)$ for appropriate λ and z we prove uniqueness of positive solutions, existence and non-existence of positive solutions, characterize the bifurcation points and show that positive solutions increase with λ and possibly other parameters. Closely related results are contained in [7] and [8].

The second order operator L is uniformly elliptic self-adjoint, say:

$$(1.2a) \quad L\varphi \equiv \sum_{i,j=1}^m \partial_i a_{ij}(x) \partial_j \varphi - a_0(x)\varphi;$$

with

$$(1.2b) \quad \sum_{i,j=1}^m a_{ij}(x) \xi_i \xi_j \geq a^2 \sum_{i=1}^m \xi_i^2, \quad a_0(x) \geq 0, \quad r(x) > 0 \quad \text{on } \bar{D}.$$

The conormal derivative $\partial/\partial \nu$ is

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