

Product Integral Formulae for Nonlinear Expansive Semi-Groups and Non- Expansive Evolution Systems

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1. Introduction. Suppose that T is a non-expansive strongly continuous one-parameter semi-group of linear transformations on the Banach space S and that A is the infinitesimal generator of T (see [6], especially Sec. 10.3, 11.6, 12.2, 12.3). One may prove first that $T(x)p = \lim_{n \rightarrow \infty} [(I - (x/n)A)^{-n}]p$ for all p in S , $x \geq 0$. Having this result one can now consider instead of a non-expansive semi-group, a strongly continuous semi-group L (with infinitesimal generator B) which for some $c > 0$ satisfies $|L(x)| \leq e^{cx}$ for all $x \geq 0$. To prove a somewhat similar exponential formula for L consider the semi-group K defined by $K(x) = e^{-cx}L(x)$ for all $x \geq 0$. K is clearly non-expansive and has infinitesimal generator $-cI + B$. Hence, if $x \geq 0$ and p is in S ,

$$\begin{aligned} e^{-cx}L(x)p &= K(x)p \\ &= \lim_{n \rightarrow \infty} [I - (x/n)(-cI + B)]^{-n}p = \lim_{n \rightarrow \infty} [(1 + (xc/n))I - (x/n)B]^{-n}p \\ &= \lim_{n \rightarrow \infty} (1 + (xc/n))^{-n} [I - ((x/n)/(1 + (xc/n)))B]^{-n}p. \end{aligned}$$

Since $\lim_{n \rightarrow \infty} (1 + (xc/n))^{-n} = e^{-xc}$ it follows that

$$\lim_{n \rightarrow \infty} [I - ((x/n)/(1 + (xc/n)))B]^{-n}p \text{ exists and is equal to } L(x)p.$$

One purpose of the present note is to extend the above development to a nonlinear setting. There are two problems in attempting such an extension. The first is that there doesn't seem to be known a general exponential formula for non-expansive nonlinear semi-groups. The second is that even in cases where one has such a formula and L is a semi-group which satisfies $|L(x)| \leq e^{cx}$

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