

On the Right Weak Solution of the Cauchy Problem for a Quasilinear Equation of First Order

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Consider the quasilinear equation of first order

$$(1) \quad u_t + f(u)_x = 0$$

with t, x, u being real scalars and $f(u)$ being a given smooth function on $(-\infty, \infty)$. Wanted is a solution $u = u(t, x)$, $t > 0$, that is prescribed for $t = 0$. The solution can be found by means of the classical method of characteristics. Such a smooth solution exists, in general, only in some neighborhood of the initial line $t = 0$, $u = u_0(x)$. Under wide assumptions about $f(u)$, however, the solution can be continued beyond this neighborhood. The continued solution has, in general, lines of discontinuity in analogy to the shock waves of compressible fluid flow. It is only a weak solution of the Cauchy problem for (1). The property of being a weak solution does not, however, characterize the function $u(t, x)$. The Cauchy problem for (1) has always more than one weak solution if $f(u)$ is not linear in $(-\infty, \infty)$. It must be remembered that the particular weak solution $u(t, x)$ spoken of above is originally thought of as

$$(2) \quad u(t, x) = \lim_{\epsilon \downarrow 0} u(t, x; \epsilon)$$

where $u(t, x; \epsilon)$ is the solution of the parabolic second order equation ($\epsilon > 0$)

$$(3) \quad u_t + f(u)_x = \epsilon u_{xx}$$

with the same initial data $u_0(x)$ at $t = 0$. This is, of course, a complicated description of the particular weak solution $u(x, t)$, and it is an important and apparently not completely solved problem to characterize this solution $u(x, t)$ in substantially simpler ways. The present note contains another approach to this question.

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