

# *On the Asymptotic Solution of a Nonlinear Dirichlet Problem*

M. S. BERGER & L. E. FRAENKEL

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In this article we study the behaviour of the positive solution of the Dirichlet problem

$$(1) \quad \begin{cases} K_\epsilon u \equiv \epsilon^2 \Delta u + u - g^2(x)u^3 = 0 & \text{in } \Omega, \\ u|_{\partial\Omega} = 0. \end{cases}$$

Here (and throughout the paper, except where the contrary is explicitly stated)  $\Omega$  is a bounded (open) domain in  $\mathbf{R}^N$ ,  $N \leq 3$ , with smooth boundary  $\partial\Omega$  of class  $C^\infty$ ; and  $g(x)$  is a smooth ( $C^\infty$ ) function, strictly positive on  $\bar{\Omega}$ . The principal results are as follows.

(A) For sufficiently small values of  $\epsilon$ , the problem (1) has a unique smooth positive solution  $u(x, \epsilon)$  which tends to  $1/g(x)$  as  $\epsilon \rightarrow 0$ , outside a narrow boundary layer of width  $O(\epsilon)$  concentrated near  $\partial\Omega$ .

(B) We construct an approximate solution  $U_M(x, \epsilon) = \sum_0^M \epsilon^m u_m(x, \epsilon)$  such that  $u(x, \epsilon) - U_M(x, \epsilon)$  is  $O(\epsilon^{M+1})$  uniformly on  $\bar{\Omega}$  (for any integer  $M$ ).

(C) The positive solution  $u(x, \epsilon)$  can be uniquely continued as a continuous function of  $\epsilon$  to the open interval  $(0, \lambda_1^{-1/2})$ , and no further. (Here  $\lambda_1$  denotes the lowest eigenvalue of  $\Delta$  in  $\Omega$ , subject to the null boundary condition, and is the value of  $\epsilon^{-2}$  at which the positive solution of (1) bifurcates from the trivial solution  $u \equiv 0$ .) This result on the existence and continuity of  $u(x, \epsilon)$  is not restricted to three space dimensions ( $N \leq 3$ ), nor to infinitely smooth domains and functions  $g$ .

## 1. INTRODUCTION

**1.1.** The results presented here are of interest in several respects. First, from the abstract point of view, the lower-order term  $u - g^2(x)u^3$  in (1) alternates in sign for  $u$  greater or less than  $1/g(x)$ , and so (1) is not immediately amenable to a general abstract study. As a consequence, the solution  $u(x, \epsilon)$  studied here differs considerably, for  $\epsilon$  sufficiently small, from the solutions with small norm that occur near points of bifurcation from the trivial solution  $u \equiv 0$ .