

Powers of Partial Isometries

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Introduction. One-parameter semigroups of unitary operators on a Hilbert space are characterized by Stone's theorem [4]; the corresponding characterization for isometries was given by Cooper [1]. Even the discrete analogues of these characterization theorems are not quite trivial; they say, respectively, that a unitary operator has a skew-Hermitian logarithm, and that an isometry is a direct sum of a unitary operator and a unilateral shift of suitable multiplicity. For partial isometries neither the continuous case nor the discrete one seems to have been studied so far. The purpose of this paper is to initiate such a study by characterizing all representations of the (discrete) semigroup of non-negative integers by partial isometries on a Hilbert space. In less ostentatious but obviously equivalent language, the problem is to characterize the operators with the property that all their positive integral powers are partial isometries.

It is convenient to have a shorter way of referring to the operators in question. We shall work throughout with a Hilbert space H , and we shall say that an operator U on H is a *power partial isometry* if U^k is a partial isometry for $k = 1, 2, 3, \dots$. Examples of power partial isometries are easy to come by. Every isometry is one, and so is every co-isometry. If an operator is both an isometry and a co-isometry, then it is unitary, and, of course, it too is a partial isometry. The elementary theory of isometries makes possible an elegant description of the power partial isometries mentioned so far: each one is a direct sum of a unitary operator, a unilateral shift (of some multiplicity), and the adjoint of a unilateral shift. It is of course understood that not all three direct summands need to be present in every case.

There are easy examples of power partial isometries that are not of the type described above, namely truncated shifts. A *truncated shift* U of index n ($= 1, 2, 3, \dots$) is obtained by considering a Hilbert space H_0 , forming the n -fold direct sum $H = H_0 \oplus \dots \oplus H_0$, and writing $U = 0$ when $n = 1$ and $U\langle f_1, f_2, \dots, f_n \rangle = \langle 0, f_1, \dots, f_{n-1} \rangle$ when $n > 1$. Every operator on a direct sum such as H can be represented as a matrix of size n whose entries are operators on H_0 ; the truncated shifts are the operators whose matrices look like the Jordan blocks