

# *Semi-Groups of Partial Isometries*

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**§1. Preliminaries and notation.** The purpose of this paper is to give a representation theorem for some one-parameter semi-groups of partial isometries on a Hilbert space  $H$ . Our model is the semi-group  $\{S_t\}$  defined on  $L_2(0, 1)$  by

$$S_t = 0 \quad \text{for } t \geq 1$$

and

$$S_t f(x) = \begin{cases} 0 & \text{for } 0 \leq x \leq t \\ f(x - t) & \text{for } t < x \leq 1. \end{cases}$$

$\{S_t\}$  is related to the classical Volterra integration operation  $J$  by  $J = \int_0^1 S_t dt$ . Our representation theorem tells when a semi-group is a direct sum of  $\{S_t\}$ 's.

**Theorem.** *Let  $\{W_t\}$  be a strongly continuous semi-group on  $H$  satisfying*

- (a) *each  $W_t$  is a partial isometry,*
- (b)  *$W_1 = 0$  and  $W_t \neq 0$  for  $0 \leq t < 1$ , and*
- (c) *the von Neumann algebra  $\mathfrak{A}$  generated by the operators  $W_t, 0 \leq t$ , is a factor.*

*Then  $\{W_t\}$  is unitarily equivalent to a direct sum of  $n$  copies ( $n$  may be infinite) of  $\{S_t\}$ .*

An immediate consequence is the

**Corollary.**  *$\{S_t\}$  is characterized up to unitary equivalence by (a), (b) and irreducibility.*

The main tool in our proof is the following consequence of [1]. Let  $W$  be a partial isometry all of whose powers are again partial isometries and which is nilpotent of index  $n$ . Then  $H = H_{n_1} \oplus H_{n_2} \oplus \cdots \oplus H_{n_k}, 1 \leq n_1 < n_2 < \cdots < n_k = n$ , each  $H_p$  ( $p = \text{some } n_i$ ) reduces  $W$ , and  $W|_{H_p}$  is unitarily equivalent to an operator (defined on the  $p$ -fold direct sum of a Hilbert space with itself) having the  $p \times p$  matrix representation