

*Differential Operators of Infinite Order on a Lie Group I^**

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Introduction. Let G be a Lie group, π a unitary representation of G , and $\partial\pi$ the associated representation of the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$. Many differential operators of interest in analysis are of the form $\partial\pi(X)$, $X \in \mathfrak{U}(\mathfrak{g})$ (e.g. all differential operators with polynomial coefficients on \mathbf{R}^n occur this way for G a suitable nilpotent Lie group). Now one of the basic problems of operator theory is the construction of functions of an operator, i.e. an "operational calculus." In the case of self-adjoint operators on Hilbert space, spectral theory provides a solution to this problem. However, Nelson and Stinespring [8] showed that an operator $\partial\pi(X)$, even if formally symmetric, may fail to have self-adjoint extensions when G is non-abelian. Thus other methods are needed to obtain a general "operational calculus" for these operators.

There is an obvious way to define certain functions of an operator, of course, independently of spectral theory. Namely, given a formal power series $f(t) = \sum_{n \geq 0} a_n t^n$ and an operator T (unbounded) on a Banach space, we may define $f(T)v = \sum a_n T^n v$, with domain $\mathfrak{D}(f(T))$ the set of vectors v in $\bigcap_{n \geq 0} \mathfrak{D}(T^n)$ for which the series is absolutely convergent. Here one has an interplay between the rate of growth of $\|T^n v\|$ and the rate of decrease of the coefficients a_n . For a given T and f , the problem is thus to construct vectors v so that $n \mapsto \|T^n v\|$ has a prescribed order of growth, depending on f (e.g. for $f(t) = e^t$, the allowable growth is $C^n n!$, $C < 1$; such a v is an analytic vector for T). In the present paper we study this problem for operators $T = \partial\pi(X)$, where π is a representation of G on a Banach space $\mathcal{H}(\pi)$ and $X \in \mathfrak{U}(\mathfrak{g})$ (we allow non-unitary representations principally to include the case of finite-dimensional representations). The paper is organized as follows:

In §1 we define a one-parameter increasing family $\mathcal{S}_\lambda(\pi)$, $0 \leq \lambda < \infty$, of locally convex spaces contained in the space $\mathcal{H}^\infty(\pi)$ of C^∞ vectors for π . These spaces are defined in terms of the order of growth of $\|\partial\pi(X_\alpha)v\|$ with $|\alpha|$ (here

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