

# Capacity and Compact Imbeddings

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**1. Introduction.** In this paper we give a condition on an open set  $G$  in Euclidean  $n$ -space  $E_n$ , which is *necessary and sufficient* for the compactness of the Sobolev space imbedding

$$(1) \quad W_0^{m,p}(G) \rightarrow L^p(G).$$

This condition is formulated in terms of a certain definition of capacity of a closed subset of a cube in  $E_n$  which is due in the case  $p = 2$  to Kondrat'ev [8] and which for sets of small capacity is equivalent, in the case  $m = 1$ , to the classical capacity of Wiener. A different generalization of the Wiener capacity ( $p = 2$ ) is used by Maz'ja [9] to study certain compact imbedding problems.

In Section 2 we define our capacity and prove the main theorem characterizing in terms of this capacity the domains  $G$  for which (1) is compact. In Section 3 we formulate a geometric condition on  $G$  which guarantees the compactness of (1). This condition is weaker than all similar geometric conditions for compactness of Sobolev imbeddings formulated by C. Clark [6] and the author [1-5].

**2. The  $C_n^{m,p}$  Capacity.** Let  $H$  be an  $n$ -dimensional cube of side  $h$  in  $E_n$ . It will be understood that any such cube referred to throughout this paper has its faces parallel to the coordinate planes. Let  $E$  be a closed subset of  $H$ ; let  $1 \leq p < \infty$  and let  $m$  be a positive integer. We denote by  $C^\infty(H, E)$  the class of all infinitely differentiable functions (real or complex valued) on  $H$  which vanish identically in a neighbourhood of  $E$ . We define the functional

$$I_{n,H}^{m,p}(u) = \sum_{1 \leq |\alpha| \leq m} h^{|\alpha|p} \int_H |D^\alpha u(x)|^p dx$$

where, as usual,  $\alpha = (\alpha_1, \dots, \alpha_n)$  is an  $n$ -tuple of non-negative integers;  $|\alpha| = \alpha_1 + \dots + \alpha_n$ ;  $D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n}$  where  $D_i = \partial/\partial x_i$ . Denoting by  $\|u\|_{0,p,A}$  the  $L^p(A)$  norm of  $u$  given by

$$\|u\|_{0,p,A}^p = \int_A |u(x)|^p dx,$$

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