

# *Invariant Subspaces and Analytic Continuation in Several Variables*

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For  $N \geq 1$  let  $U^N$  denote the unit polydisc in  $N$  complex variables and  $H^2(U^N)$  the usual Hardy class of holomorphic functions in  $U^N$ . A closed subspace  $M$  of  $H^2(U^N)$  is said to be invariant if it is invariant under multiplication by polynomials. If  $N = 1$ , then any invariant subspace is of the form  $f_0 H^2(U^1)$  where  $f_0$  is an inner function [2], and the following conditions are equivalent for a point  $\lambda \in \partial U^1$ .

- (a)  $f_0$  has an analytic continuation into a neighborhood of  $\lambda$ .
- (b)  $I - \bar{\lambda}T$  is invertible, where  $Tf = P_{M^\perp}(zf)$  for  $f \in M^\perp = H^2 \ominus f_0 H^2$ .
- (c) Every function in  $M^\perp$  has an analytic continuation into a neighborhood of  $\lambda$ .

The equivalence of (a) and (b) was proved by Moeller [4]. Akutowicz and Carleson proved the equivalence of (a) and (c) in case  $f_0$  is a Blaschke product [1]. Helson, [3], extended the result to general  $f_0$  and gave a new proof of the equivalence of all three conditions.

For  $N > 1$  there is, in general, no inner function associated with an invariant subspace. In I we discuss the general problem of analytic continuation of functions in  $M^\perp$ , and the relation between the analogues of (b) and (c) for  $N > 1$ . In II we consider the case where  $M = f_0 H^2(U^N)$  where  $f_0$  is inner. In III we consider another special kind of invariant subspace and obtain a theorem of Akutowicz–Carleson type. This class of subspaces includes, in particular, all invariant subspaces of finite codimension, as we show in IV.

I. In what follows  $M$  will denote an invariant subspace of  $H^2 = H^2(U^N)$  and  $M^\perp$  will denote its orthogonal complement in  $H^2$ . We will consider the

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