

On a Conjecture of Levi-Civita

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1. Introduction. In 1925 Levi-Civita [5] reduced the mathematical theory of two-dimensional, periodic, permanent waves of finite amplitude on an ideal fluid of infinite depth to the problem of determining all non-constant complex valued functions $\omega(\zeta) = \theta + i\tau$ of a complex variable $\zeta = \rho e^{i\sigma} = \xi + i\eta$ holomorphic in the unit disk $|\zeta| \leq 1$ which satisfy $\omega(0) = 0$ and

$$(1.1) \quad \partial\theta/\partial\rho = p e^{-3\tau} \sin\theta \quad \text{on} \quad |\zeta| = 1,$$

where p is an *a priori* undetermined positive constant. It is assumed that $\omega(\zeta) \in C'(|\zeta| \leq 1)$ and without loss of generality that ω is to be purely imaginary for real ζ [5]. This problem will be referred to as problem (I).

Each such function ω for which $|\omega|$ is sufficiently small gives rise to a periodic wave (and conversely) where θ is the angle of inclination of the flow vector with the horizontal and τ is related to the speed q of the flow vector, *viz.*, $\tau = \log(q/c)$ where c is the speed of the wave profile. The constant p is related to physical constants by

$$(1.2) \quad p = \frac{g\lambda}{2\pi c^2},$$

where λ is a wavelength of the wave and g is the acceleration due to gravity.

One is interested in the existence of waves in a neighborhood of the uniform flow $\omega \equiv 0$ and, consequently, requires that $|\omega|$ be small. Specifically it is required that

$$(1.3) \quad |e^{-i\omega} - 1| \leq \beta < 1, \quad |\zeta| \leq 1,$$

for some constant β , $0 < \beta < 1$. Using the technique of analytic expansion in powers of a small parameter (essentially the amplitude of the wave), Levi-Civita demonstrates the existence of a non-constant solution ω to problem (I) satisfying (1.3) for any value of p satisfying $1 - \epsilon \leq p \leq 1$ where $\epsilon > 0$ is some sufficiently small constant. The result has been proved also by Littman and Nirenberg (see Stoker [9]) using a fixed point theorem and by Nekrasov [8] and Lichtenstein [6] using the theory of integral equations. Each such solution to problem (I) gives rise to a unique wave motion as soon as a wavelength λ is specified; c is then determined from (1.2).