

# Nonlinear Parabolic Initial-Value Problems

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Nonlinear parabolic initial-value problems have been studied by Browder [1] using the theory of monotone operators involving a densely defined operator in a Banach space  $X$  and by Lions [3] using monotone operators defined on all of  $X$  together with the method of elliptic regularization. The above methods, in spite of their generalities, cannot be applied to the study of the non-stationary Navier-Stokes Equations.

The purpose of this paper is to extend the results of [1], [3] (in the case when  $X$  is a Hilbert space) so as to include the Navier Stokes Equations. The method used is based on that of Lions [2].

The hypotheses are given in Section 1. An existence theorem is proved in Section 2. Applications are given in Section 3. First, a parabolic equation whose elliptic part is of divergence form  $A(u) = \sum_{|\alpha| \leq m} D^\alpha(x, t, u, \dots, D^m u)$  with  $A_\alpha(x, t, \xi_1, \dots, \xi_m)$  having at most a linear growth in  $(\xi_1, \dots, \xi_m)$  is considered. Then the generalized Navier-Stokes Equations are studied. The usual Laplacian in the equations is replaced by a nonlinear elliptic operator of divergence form and the space-domain is a bounded open subset of  $R^n$  with  $n \leq 4$ .

**Section 1** Let  $V$  and  $H$  be two real Hilbert spaces with a continuous natural injection mapping of  $V$  into  $H$  with  $V$  dense in  $H$ . Let  $(\cdot, \cdot)$  denote the inner product in  $H$  and  $((\cdot, \cdot))$  denote the pairing between  $V$  and its dual  $V^*$ .

Consider the Hilbert space  $L^2(0, T; V)$  of equivalence classes of functions  $v$  from  $[0, T)$  to  $V$  with  $\int_0^T \|u(t)\|_V^2 dt < \infty$ . The norm in this Hilbert space is

$$\|u\|_{L^2(0, T; V)} = \left\{ \int_0^T \|u(t)\|_V^2 dt \right\}^{1/2}.$$

Similarly, we define  $L^2(0, T; H)$ . Let  $g \in L^2(0, T; V^*)$ ,  $f \in L^2(0, T; V)$ ; then  $\langle g, f \rangle$  denotes the pairing between  $L^2(0, T; V^*)$  and  $L^2(0, T; V)$ . If  $g \in L^2(0, T; H)$  and  $f \in L^2(0, T; V)$ , then

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