

# On the Commutativity of Homotopy Groups

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Let  $[X, Y]$  denote the set of based homotopy classes of continuous maps from  $X$  to  $Y$ . We assume all spaces have base points and use  $*$  to denote both the base point and the constant map. The reduced suspension of  $X$  is denoted by  $\Sigma X$  and the loop space of  $X$  by  $\Omega X$ . All spaces are assumed to be connected.

The one point union,  $X \vee Y$ , is the subset of  $X \times Y$  consisting of points with at least one coordinate equal to the base point. The folding map,  $\nabla: X \vee X \rightarrow X$  is defined by  $\nabla(x, *) = \nabla(*, x) = x$ .  $X$  is called an  $H$ -space if  $\nabla$  can be extended to  $X \times X$ . Such an extension is called a multiplication on  $X$ . Dually  $Y$  is called an  $H'$ -space if there is  $\varphi: Y \rightarrow Y \vee Y$  such that  $\varphi$  followed by inclusion into the product is homotopic to the diagonal,  $\Delta: Y \rightarrow Y \times Y$ . In this case  $\varphi$  is called a comultiplication. Composition of loops gives  $\Omega X$  an  $H$ -space structure and the "pinching map" gives  $\Sigma Y$  an  $H'$ -space structure. These structures are taken as canonical.

Since  $[\Sigma X, Y] \approx [X, \Omega Y]$  it is easily seen that  $[\Sigma X, Y]$  is an abelian group if  $\Omega Y$  is homotopy-abelian and dually  $[X, \Omega Y]$  is an abelian group if  $\Sigma X$  is homotopy-abelian. We show that the first result holds if  $\Sigma X$  is replaced by an arbitrary  $H'$ -space and dually the second result holds if  $\Omega Y$  is replaced by an arbitrary  $H$ -space.

**Theorem.**  $[X, Y]$  is an abelian group if either

- (a)  $X$  is an  $H'$ -space and  $\Omega Y$  is homotopy abelian, or
- (b)  $Y$  is an  $H$ -space and  $\Sigma X$  is homotopy abelian.

*Proof.* Let  $e: \Sigma \Omega X \rightarrow X$  and  $\lambda: Y \rightarrow \Omega \Sigma Y$  be the adjoints of  $1_{\Omega X}$  and  $1_{\Sigma Y}$  respectively. If  $X$  is an  $H'$ -space,  $e^*: [X, Y] \rightarrow [\Sigma \Omega X, Y]$  is one-one. Dually, if  $Y$  is an  $H$ -space,  $\lambda_*: [X, Y] \rightarrow [X, \Omega \Sigma Y]$  is one-one. In both cases the range is an abelian group. To prove the theorem it therefore suffices to show that in case (a),  $e^*$  is a homomorphism, and in case (b),  $\lambda_*$  is a homomorphism.

We first prove (a). Let  $\bar{\varphi}: X \rightarrow X \vee X$  and  $\varphi: \Sigma \Omega X \rightarrow \Sigma \Omega X \vee \Sigma \Omega X$  be the comultiplications in  $X$  and  $\Sigma \Omega X$  respectively.  $e^*$  is a homomorphism if and only

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