

On the Fourier Coefficients of Outer Functions

T. L. KRIETE III

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Introduction. In this note we consider the problem raised by Helson [3, p. 22] of identifying those sequences which can arise as

$$\sum_{j=n}^{\infty} |a_j|^2, \quad n = 0, 1, 2, \dots,$$

where $\{a_j\}$ are the Fourier coefficients of an outer function in H^2 . We obtain a lower estimate for such sequences and use it to show that they can vanish arbitrarily slowly as $n \rightarrow \infty$. The proof is applied to answer a question about cyclic vectors of the backward shift on H^2 (see the question raised after Proposition 2). Our methods are elementary. The interested reader can rephrase Proposition 1 and Theorem 1 in terms of linear prediction theory; see, e.g., [2, Chapter 10].

I wish to thank Alan Zame for supplying the important part of the proof of Lemma 1.

First we need some notation. C is the unit circle in the complex plane and σ is normalized Lebesgue measure on C (or, sometimes, on $[0, 2\pi)$). χ is the identity function on C : $\chi(e^{ix}) = e^{ix}$. The value of a function f at e^{ix} will be written $f(e^{ix})$ or $f(x)$.

For $1 \leq p \leq \infty$, L^p and H^p denote the Lebesgue and Hardy spaces with respect to σ ; the relevant information about these spaces can be found in [3] or [4]. If $0 \leq w \in L^1$, $G(w)$ is the geometric mean of w :

$$G(w) = \exp \int \log w \, d\sigma,$$

where the right side is taken to be zero if $\log w$ is not integrable. $G(w)$ never exceeds the arithmetic mean $\int w \, d\sigma$.

A function f in H^2 is *outer* provided $\{\chi^n f: n \geq 0\}$ spans H^2 ; $g \in H^2$ is *inner* if $|g| = 1$ a.e. It is well known that $\log |f| \in L^1$ (i.e., $G(|f|) > 0$) whenever $f \in H^1$ and f is not the zero function. If $0 \leq w \in L^1$, then in order for $G(w)$ to be positive, it is necessary and sufficient that $w = |f|^2$ a.e. for some outer function f .

For any positive integer n , let

$$\rho_n(w) = \inf \int |1 - \chi^n p|^2 w \, d\sigma,$$