

Decomposition of Semi-Groups of Partial Isometries

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Communicated by P. R. HALMOS

§1. Introduction. The Jordan canonical form tells us that, given a nilpotent operator N on a finite dimensional Hilbert space F , we can find a new inner product on F such that N^k is a partial isometry for all k (*i. e.* N is a power partial isometry in the terminology of [1]). Moreover, in this new inner product, F splits into a direct sum of subspaces F_k , on each of which N acts as a truncated shift (see [1]) of index, say, i_k . That is, $F_k = \sum^{i_k} \oplus G$ and $N|_{F_k}$ has the matrix representation

$$\left. \begin{pmatrix} 0 & \cdot & \cdot & \cdot & 0 \\ I_G & 0 & \cdot & \cdot & 0 \\ \vdots & & & & \vdots \\ \vdots & & & & \vdots \\ 0 & \cdot & \cdot & \cdot & I_G & 0 \end{pmatrix} \right\} i_k$$

I_G being the identity on G .

All this is, of course, known to every undergraduate mathematician, even though the terminology and point of view may seem a trifle strange. In this paper, we deal with an analogous situation in Hilbert space. For a nilpotent operator, we substitute a nilpotent semi-group of operators. We do make, however, an enormous simplification by assuming from the outset that our semi-groups are composed of partial isometries. In other words, we don't consider at all the similarity theory of such semi-groups.

It turns out that the decomposition, described above in the finite dimensional case, has a quite reasonable analog in the separable case. The space decomposes into a direct integral of Hilbert spaces and the semi-group decomposes into a direct integral of semi-groups of partial isometries. Each differential semi-group has a certain index which is a real number and, most important, is a truncated shift. All these terms will be made precise below.

* Work supported by NSF grant GP-9523.