

A Sharp Maximum Principle for Degenerate Elliptic-Parabolic Equations

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For linear second order elliptic or parabolic equations, the “weak” maximum principle and the “strong” maximum principle are well known. In this paper, we introduce the *sharp* maximum principle for equations of degenerate elliptic-parabolic type; that is, for operators of the form

$$Lu = a^{ij}(x)u_{x_i x_j} + b^i(x)u_{x_i}, \quad a^{ij}(x)\xi_i \xi_j \geq 0.$$

Here the coefficients are smooth functions of $x = (x_1, x_2, \dots, x_n)$ in a connected open set $\omega \subset \mathbb{R}^n$; we use summation convention ($i, j = 1, 2, \dots, n$). Our exact assumptions will be described more precisely later on.

The sharp maximum principle deals with the following question: Suppose a point $P \in \omega$ is given. Then is it always possible to determine a subset $S(P; \omega)$ of ω such that, if $Lu \geq 0$ in $S(P; \omega)$ and u has a local maximum at P , relative to the set $S(P; \omega)$, then necessarily $u \equiv u(P) = \text{const.}$ throughout its closure $\bar{S}(P; \omega)$? The answer is yes; the sharp maximum principle gives a precise description of $S(P; \omega)$.

In connection with $S(P; \omega)$ (or its closure $\bar{S}(P; \omega)$), which we call the propagation set of P in ω , several interesting phenomena arise: For either elliptic or parabolic equations, the propagation set is well known [11], [13]; its determination in a neighborhood of P is a purely local question. But for general degenerate elliptic-parabolic equations, the propagation set has a global character. For example, a small perturbation of the boundary of the region ω , far away from P , can cause a drastic change in $S(P; \omega)$, even in a neighborhood of P . Also, a small perturbation of the coefficients of L can lead to a drastic change in $S(P; \omega)$. These global phenomena come about because $S(P; \omega)$ is determined by the trajectories of certain vector fields associated with the operator L . In the familiar cases of elliptic or parabolic equations, the global behavior of these trajectories is extremely simple. However, in the general case, their global behavior is nontrivial and can change drastically with respect to the perturbations mentioned above.

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