

# *A Vietoris-Begle Theorem for Submersions*

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**1. Introduction.** Among existing versions of the Vietoris-Begle Mapping Theorem the sheaf theoretic version, as given in G. E. Bredon's *Sheaf Theory* [1], is particularly germane to the considerations of the present paper. By a slight restriction of generality this result may be stated in the following form.

**Vietoris-Begle Theorem.** *Let  $X$  be completely paracompact,  $f: X \rightarrow Y$  a closed surjective map and  $\mathcal{A}$  a sheaf of modules on  $Y$ . For all  $y \in Y$  let  $f^{-1}(y)$  be connected and let  $H^i(f^{-1}(y); \mathcal{A}_y) = 0$  for  $0 < i \leq p$ . The induced map*

$$f^!: H^i(Y; \mathcal{A}) \rightarrow H^i(X; f^* \mathcal{A})$$

*is then an isomorphism for  $0 \leq i \leq p$  and a monomorphism on dimension  $p + 1$ .*

Here  $\mathcal{A}_y$  denotes the stalk of  $\mathcal{A}$  at  $y$  (considered as a sheaf on  $X$ ),  $f^* \mathcal{A}$  denotes the inverse image of  $\mathcal{A}$  under  $f$  and  $H^i$  denotes sheaf cohomology. For basic facts, terminology and notations relating to sheaf theory we refer to Bredon's text.

The condition that  $f$  be closed is quite indispensable, unless it is assumed that  $f$  admits some other nice properties. In the present paper we shall be concerned with the case where  $X$  and  $Y$  are differentiable manifolds (of class  $C^\infty$ , let us say) and  $f: X \rightarrow Y$  is a *submersion*, i.e. a  $C^\infty$ -map whose differential is everywhere surjective. More precisely, we shall take  $X$  to be open and paracompact but will allow  $Y$  to be an arbitrary (non-Hausdorff)  $C^\infty$ -manifold. One observes that a surjective submersion  $f: X \rightarrow Y$  whose fibres  $f^{-1}(y)$  are connected is now precisely equivalent to the natural projection of  $X$  onto the quotient space induced by a regular foliation on  $X$  (for basic facts pertaining to foliations we refer to Palais [3], Chapter I). The question is whether one obtains a Vietoris-Begle Theorem for this class of maps without the requirement that  $f$  be closed. The main result of this paper (Theorem 1) provides an affirmative answer. However, the resulting Vietoris-Begle Theorem is slightly weaker than its classical counterpart in two respects. Firstly, the acyclicity condition on the

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