

*Multiplication Operators on Incidence Algebras**

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1. Introduction. If \mathcal{A} is an algebra over a field K , (left or right) multiplication by an element of \mathcal{A} is a linear operator on \mathcal{A} . This assertion is completely trivial, of course, but there are situations in which it is convenient, even interesting, to consider multiplication by a fixed element as an operator.

For example, if \mathcal{A} is the incidence algebra of a locally finite partially ordered set P , then multiplications by ζ (the characteristic function of the ordering) and μ (the Möbius function) are, respectively, summation and differencing operators, and the Möbius Inversion Formula is the “fundamental theorem of (difference) calculus” on P .

In this paper we consider multiplication operators on an incidence algebra induced in a natural way by any subset of the partially ordered set, by a closure operator on the set, by a Galois connection (residuated mapping) between two partially ordered sets, and by an increasing function from one set to another.

The principal results concerning these operators, while basically very simple, provide simplifications and generalizations of some results of Rota and Crapo on Möbius inversion. The fundamental idea of studying Möbius inversion across a Galois connection is due to Rota [8], and the present work in part continues the refinement of this idea by Crapo [4]. There are also some generalizations of results on arithmetic functions, and in this sense this paper is a sequel to the author’s series of papers on that subject [12, 13, 14]. Some of the results have combinatorial interest, as indicated briefly in §4.

Most of the operators considered in this paper are multiplications by idempotent elements, and hence are Baxter operators. In an Appendix, we give several necessary and sufficient conditions for an averaging, Baxter operator on an associative algebra with 1 to be a multiplication by an element of the algebra.

2. Multiplication operators induced by a subset. Let (P, \leq) be a locally finite partially ordered set and $\mathcal{A} = \mathcal{A}(P)$ its incidence algebra over a field K .

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