

# Toeplitz Operators with Locally Sectorial Symbols

R. G. DOUGLAS<sup>1</sup> & HAROLD WIDOM<sup>2</sup>

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Let  $L^2$  denote the complex Lebesgue space for normalized Lebesgue measure on the unit circle  $\mathbf{T}$ ,  $H^2$  the associated Hardy space of functions with zero negative Fourier coefficients, and  $P$  the projection of  $L^2$  onto  $H^2$ . For  $\varphi$  in  $L^\infty$  we define the Toeplitz operator  $T_\varphi$  on  $H^2$  by  $T_\varphi f = P(\varphi f)$  for  $f$  in  $H^2$ . The operator  $T_\varphi$  is clearly bounded and the mapping  $\varphi \rightarrow T_\varphi$  is linear, preserves adjoints, and is isometric. The mapping is not, in general, multiplicative, and it is this fact which makes the problem of deciding which  $T_\varphi$  are invertible both interesting and nontrivial. In this note we give a purely local condition on the function  $\varphi$  from which one can deduce that the Toeplitz operator  $T_\varphi$  is a Fredholm operator and can determine by inspection whether it is invertible.

Our results generalize those previously obtained by Widom [5] and Devinatz [2] for the case of piecewise continuous functions.

A measurable function is called *sectorial* if its essential values lie in an open sector of opening less than  $\pi$  having its vertex at the origin and *locally sectorial* if every point in the domain has a neighborhood on which the function is sectorial. We shall show that if  $\varphi$  is bounded away from zero and is locally sectorial, then  $T_\varphi$  is a Fredholm operator. We shall also determine the index of  $T_\varphi$  (that is, the dimension of its null space minus the codimension of its range).

The requirement that  $\varphi$  be bounded away from zero is necessary for  $T_\varphi$  to be a Fredholm operator. (This is essentially due to Hartman and Wintner [3].) The condition that  $\varphi$  be locally sectorial, although not necessary, seems to be the most general local sufficient condition one can find.

The crucial fact is that it is possible to assign an index (the winding number about the origin for a non-vanishing continuous function) to any locally sectorial

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