

# *An Application of the Holomorphic Functional Calculus to Sharpen a Classical Matrix Commutator Theorem*

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1. In [3] (Theorem 1), M. Marcus and R. C. Thompson prove the following fact.

*Theorem I.* Suppose that  $A$  and  $B$  are operators on a finite dimensional Hilbert space. Assume that

- (i)  $0 \notin \sigma(A)$  (spectrum  $A$ ) and  $0 \notin$  numerical range  $B$ ;
- (ii)  $ABA^{-1}B^{-1}$  commutes with  $A$ ;
- (iii)  $A$  and  $ABA^{-1}B^{-1}$  are normal.

Then  $B$  commutes with  $A$ .

As the authors point out, this is an improvement of a classical theorem.

On the other hand, in [2] (page 21), I. Kaplansky has shown the following fact.

*Theorem II.* Suppose that  $A$  and  $C$  are operators on a Hilbert space. Assume that

- (ii)'  $CA - AC$  commutes with  $A$ ;
- (iii)'  $A$  is normal.

Then  $C$  commutes with  $A$ .

In this note we shall use Theorem II and the holomorphic functional calculus (see [1], sections 5.2, 5.3, 5.4) to prove the following result.

*Theorem III.* Suppose that  $A$  and  $B$  are operators on a Hilbert space. Assume that

- (i)'  $0 \notin$  convex hull  $\sigma(B)$ ;
- (ii)  $BAB^{-1}$  commutes with  $A$ ;
- (iii)'  $A$  is normal.

Then  $B$  commutes with  $A$ .

As a consequence, hypotheses (i) and (iii) of Theorem I may be weakened to (i)' and (iii)' of Theorem III.

Whereas Theorem I is proved by a simple matrix computation, Theorem III