

# *On the Normalization of Haar Measures in the Representation Theory of Compact Semi-Simple Lie Groups*

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**1. Introduction.** Let  $\mathfrak{g}$  be a compact real semi-simple Lie algebra,  $\mathfrak{j}$  a Cartan subalgebra of  $\mathfrak{g}$ ; let  $\mathfrak{g}_\mathbb{C}$  (respectively  $\mathfrak{j}_\mathbb{C}$ ) denote the complexification of  $\mathfrak{g}$  (respectively  $\mathfrak{j}$ ); let  $\Phi = \{\alpha\}$  denote the set of roots of the pair  $(\mathfrak{g}_\mathbb{C}, \mathfrak{j}_\mathbb{C})$ ,  $\Phi^+$  the positive roots (relative to some ordering of  $\Phi$ ); let  $W$  denote the Weyl group of the pair  $(\mathfrak{g}_\mathbb{C}, \mathfrak{j}_\mathbb{C})$ . Because  $\mathfrak{g}$  is compact, the Killing form  $B$  of  $\mathfrak{g}$  is negative definite; agreeing to use  $-B$ , we see, therefore, that  $\mathfrak{g}$  may be equipped with the structure of a Euclidean space in a natural way. It is well-known that the restriction of  $B$  to  $\mathfrak{j}$  is non-degenerate; therefore  $\mathfrak{j}$  may also be regarded as a Euclidean space in the obvious manner.

Now suppose that  $G$  is a connected Lie group with Lie algebra  $\mathfrak{g}$ ; let  $J$  denote the analytic subgroup of  $G$  with Lie algebra  $\mathfrak{j}$ —then  $J$  is a maximal torus in  $G$ . The group  $G$  operates on  $\mathfrak{g}$  in the usual way (namely through the adjoint representation); we shall agree to write  $xX$  in place of  $Ad(x)X$  ( $x \in G, X \in \mathfrak{g}$ ). Let  $d_\mathfrak{g}(X)$  (respectively  $d_\mathfrak{j}(H)$ ) denote the canonical element of volume on  $\mathfrak{g}$  (respectively  $\mathfrak{j}$ ) as measured by  $-B$ ; let  $d_G(x)$  denote normalized Haar measure on  $G$  (hence  $\int_G d_G(x) = 1$ ); let  $\pi$  denote the polynomial function on  $\mathfrak{j}_\mathbb{C}$  which is defined by the requirement  $\pi(H) = \prod_{\alpha > 0} \alpha(H)$  ( $H \in \mathfrak{j}_\mathbb{C}$ )—then, according to Harish-Chandra [1, p. 105], there exists a positive constant  $\text{vol}(G)$  such that, for all continuous functions  $f$  on  $\mathfrak{g}$  with compact support, we have

$$\int_{\mathfrak{g}} f(X) d_\mathfrak{g}(X) = \text{vol}(G) \int_{\mathfrak{j}} \left| \pi(H) \right|^2 d_\mathfrak{j}(H) \int_G f(xX) d_G(x).$$

The objective of the present note is to compute  $\text{vol}(G)$ .

Let  $r$  denote the cardinality of the set  $\Phi^+$ ,  $[W]$  the order of the Weyl group  $W$ ; let  $(\cdot, \cdot)$  denote the usual scalar product on the real span (in the dual of  $\mathfrak{j}_\mathbb{C}$ ) of the elements  $\alpha$  in  $\Phi$ ; let  $\rho$  be one-half the sum of the positive roots (i.e. let  $\rho = 2^{-1} \sum_{\alpha > 0} \alpha$ )—then it turns out that

$$\text{vol}(G) = (2\pi)^r / ([W] \prod_{\alpha > 0} (\rho, \alpha)) \quad (\pi = 3.14 \dots).$$