

The Weyl Spectrum of an Operator*

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1. Introduction. The concept of Weyl spectrum is native to the theory of perturbation of the spectrum, but recently it has found application to other themes in operator theory. The aims of this article are to give a brief exposition of Weyl spectrum for (bounded linear) operators in Hilbert space, discuss its spectral mapping properties, show its coincidence with the set of “limit points” of the spectrum (suitably defined) for some classes of not necessarily normal operators, and exhibit some applications to polynomially compact operators, commutators, and spectral theory in the Calkin algebra.

In what follows, \mathcal{H} denotes an infinite-dimensional Hilbert space (the theory of Weyl spectrum is vacuous for finite-dimensional spaces). If T is an operator, we write $N(T)$ and $R(T)$ for the null space and range of T ; $\sigma(T)$ for the spectrum of T ; $\pi_0(T)$ for the set of eigenvalues of T ; $\pi_{0,r}(T)$ for the eigenvalues of finite multiplicity; $\pi_{0,i}(T)$ for the eigenvalues of infinite multiplicity; and $\pi_{0o}(T)$ for the isolated points of $\sigma(T)$ that are eigenvalues of finite multiplicity.

If \mathcal{I} is the class of invertible operators, then $\sigma(T) = \{\lambda: T - \lambda I \notin \mathcal{I}\}$. Analogously, let

$$\sigma_{\mathcal{F}}(T) = \{\lambda: T - \lambda I \notin \mathcal{F}\},$$

where \mathcal{F} is the class of all Fredholm operators. {An operator T is called a *Fredholm operator* if $N(T)$ is finite-dimensional, $R(T)$ is closed, and $R(T)^\perp$ is finite-dimensional.} It is a theorem of F. V. Atkinson that an operator is Fredholm iff it is invertible modulo the ideal of compact operators iff it is invertible modulo the ideal of operators of finite rank [14, Problem 142]. Thus, if $L(\mathcal{H})$ denotes the algebra of all operators, and $L_c(\mathcal{H})$ the ideal of all compact operators, and if $\hat{T} = T + L_c(\mathcal{H})$ denotes the canonical image of T in the quotient Banach algebra $L(\mathcal{H})/L_c(\mathcal{H})$ (which is a C^* -algebra [11, p. 17], called the *Calkin algebra* of \mathcal{H} , denoted briefly L/L_c), then by Atkinson's theorem,

$$(1) \quad \sigma_{\mathcal{F}}(T) = \{\lambda: \hat{T} - \lambda \hat{I} \text{ is not invertible in } L/L_c\} = \sigma(\hat{T});$$

in particular, $\sigma_{\mathcal{F}}(T)$ is nonempty and compact [14, Problem 73].

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