

A Real Variable Lemma and the Continuity of Paths of Some Gaussian Processes

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Introduction. The now standard method introduced by Doob of constructing a separable and measurable model for a mean continuous stochastic process, starting from a given consistent system of joint distributions, although somewhat arbitrary, is perhaps unavoidable in the general case.

However, for Gaussian processes one would hope for a more direct approach. A procedure, frequently used by communications engineers in the Gaussian case, is to expand the paths in terms of the eigenfunctions of the covariance kernel. The resulting expression, which is usually referred to as the Karhunen-Loève expansion and was apparently introduced quite early by M. Kač, is a very natural tool to use.

The purpose of this paper is to show that, at least in the cases when the paths are known to be almost surely continuous this procedure can indeed be used to produce the desired models. In fact, it will also be shown that the best possible estimates for the modulus of continuity of the sample paths can be directly obtained from a study of the partial sums of this expansion.

Our main tool here is a real variable lemma whose significance will, perhaps, transcend the applications that have led us to its discovery. This is a lemma which, roughly speaking, states that the finiteness of a certain integral involving a given function has a bearing on the modulus of continuity of such a function.

Results of similar nature have appeared in Fourier analysis and partial differential equations, but as far as we know, no result of the generality of our Lemma 1.1 has appeared in the literature before.

As we shall see, the power of our lemma lies in the fact that it provides a step which enables us to pass from global estimates, often readily available in a probabilistic setting, to local estimates which usually appear to be of a more elusive nature.

1. The Lemma. Here and in the following, $\Psi(u)$ will denote a non-negative even function on $(-\infty, +\infty)$ and $p(u)$ a non-negative even function on $[-1, 1]$.