

Linear Endomorphisms of Function Algebras and Related Functional Equations

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Communicated by MAURICE HEINS

1. Introduction. The First Order Linear Functional Equation is

$$(1.1) \quad (f \circ \omega)(P) + g(P)f(P) = h(P).$$

Here, $P \in S$ where S is an arbitrary set; ω a (known) mapping of S into itself; g, h (known) and f (unknown) are mappings from S into a suitable algebra, often the reals or the complex numbers.

(1.1), which was first discussed, in a more special form, by Kordylewski and Kuczma ([1], p. 46) plays a part analogous to that of the first order linear differential equation. The operator appearing in (1.1) is

$$(1.2) \quad (\Omega f)(P) = (f \circ \omega)(P).$$

It is readily seen to be a linear endomorphism of the function algebra on which it acts (multiplication in the algebra being defined as the “pointwise” multiplication); we shall occasionally refer to Ω as a “substitution operator”, or as a “scale transformation”. The first name refers to the fact that Ω effects a “substitution” *i.e.* replaces the independent variable by a (known) function thereof; while the second name is used in mathematical physics. One observes that a mapping ω of the “underlying set” S into itself induces a linear endomorphism on the algebra of functions defined on S . It is one of the aims of the present paper to investigate linear endomorphisms of function algebras not induced by a mapping of the underlying set into itself.

The study of linear endomorphisms of function algebras is relevant to the study of (1.1), in particular if we replace (1.1) by the more general equation

$$(1.3) \quad (\Omega f)(P) + g(P)f(P) = h(P),$$

where Ω is now any linear endomorphism of a function algebra containing f . The reason for this generalization is the following. Let φ be an eigenfunction of Ω with eigenvalue λ :

* Work supported in part by the National Science Foundation, under Grant Nr. GP 11764.