

Polynomial Estimates for Betti Numbers of Loop Spaces of Symmetric and Close-to-Symmetric Riemannian Manifolds*

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The study of loop spaces and relative loop spaces of manifolds is of interest in homotopy theory and other branches of topology, and Morse theory furnishes a powerful tool for the study of loop spaces, especially if the base spaces have a lot of symmetry. By using Morse theory, Bott discovered a simple inequality which allows sums of Betti numbers of loop spaces to be estimated in terms of the volumes of certain tubular sets in the base spaces. The estimates are obtained by studying the growth of Jacobi fields along geodesics in the base manifold. Jacobi fields satisfy the Jacobi differential equation, a second-order linear differential equation whose coefficients can be expressed in terms of the curvature of the Riemannian metric on the base manifold. These estimates, developed successively in [1], [7], and [6], have been expressed in terms of bounds on the sectional curvatures and show that the sum of the Betti numbers of the loop space in dimensions at most λ is at most of exponential order in λ . If we choose to condition only the sectional curvatures, we can expect no better estimates, as the following simple example shows.

Let $a: \mathbf{R} \rightarrow \mathbf{R}$ be the function defined by

$$a(t) = \begin{cases} 4 & \text{if } 0 \leq t < \frac{\pi}{4} \text{ or } \frac{3\pi}{4} \leq t < \pi, \\ 1 & \text{if } \frac{\pi}{4} \leq t < \frac{3\pi}{4} \text{ or } \pi \leq t < \frac{5\pi}{4}, \end{cases}$$

and by $a(t) = a(t + (3\pi/2))$ for all t . The scalar differential equation

$$y''(t) + a(t)y(t) = 0,$$

has the property that every nonzero solution has exponential growth or expo-

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