

Boundaries for Natural Systems

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1. Introduction. We are concerned here mainly with a comparison of the Šilov boundary for natural systems defined by Rickart [7] and the type II boundary for locally m -convex algebras defined by the author [1]. We also relate these boundaries to the strong boundary defined for certain natural systems by Meyers [4]. Our main results are found in Section 3 where we show that for every natural system (X, A) Rickart's boundary $S_r(X, A)$ is contained in the type II boundary $S_2(X, A)$ and that when X is locally compact they agree. Further, if X is also σ -compact and if A is closed in $C(X)$, then the strong boundary $S_{st}(X, A)$ is dense in $S_r(X, A) = S_2(X, A)$. We give examples to show that in general we may have (1) $\phi = S_{st}(X, A) \neq S_r(X, A) = S_2(X, A)$; (2) $\phi = S_{st}(X, A) = S_r(X, A) \neq S_2(X, A)$; and (3) $\phi = S_{st}(X, A) \subsetneq S_r(X, A) \subsetneq S_2(X, A)$.

In [2] we showed that if A is a singly-generated (singly-rationally-generated) complete locally m -convex algebra with identity, $M(A)$ is the spectrum of A , and if $S_2(M(A), A) = \phi$, then $A = \text{Hol}(D) \oplus R(A)$, where D , the spectrum of any generator of A , is an open polynomially convex subset of \mathbf{C} (open subset of \mathbf{C}) and $R(A)$ is the radical of A . We extend this result here to show that if $S_r(M(A), A) = \phi$ (where we have restricted attention to locally m -convex algebras A for which $(M(A), A^\wedge)$ is natural), then, in fact, $M(A)$ is locally compact. Thus, $S_2(M(A), A) = \phi$ and the conclusion of the above-stated theorem is obtained. We give an example showing that $S_{st}(M(A), A) = \phi$ is not a sufficient condition in the singly-rationally-generated case.

We close by considering natural systems (X, A) where X is an open (locally compact) subset of \mathbf{C} and giving conditions in order that A consist of analytic functions on X (functions analytic in the interior of X). The results of this section are essentially corollaries of theorems of Rudin [9] or Meyers [4] and the theorems of section 2 of this paper.

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