

Operators with Compact Imaginary Part

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Let B be a C^* -algebra with identity e and let S be a linear subspace of B such that e belongs to S . An irreducible $*$ -representation π of B on a Hilbert space \mathfrak{H} is called a *boundary representation* for S if the restricted map $\pi|_S : S \rightarrow \mathcal{L}(\mathfrak{H})$ has a *unique* completely positive linear extension to B (namely π). Boundary representations were introduced in [1]; their usefulness for studying non self-adjoint families of operators derives from the following

Implementation theorem *Let S_i be a linear subspace of a C^* -algebra B_i , $i = 1, 2$, such that the identity of B_i belongs to S_i , and such that S_i generates B_i as a C^* -algebra. Assume that the intersection of the kernels of all boundary representations of B_i for S_i is $\{0\}$, $i = 1, 2$. Then every completely isometric linear map of S_1 on S_2 , which preserves identities, is implemented by a $*$ -isomorphism of B_1 onto B_2 .*

Here is one way that kind of situation can occur in operator theory. Suppose we are given a finite set $\{T_1, \dots, T_n\}$ of operators on a Hilbert space \mathfrak{H} which is irreducible in the sense that the only subspaces of \mathfrak{H} invariant under the set $\{T_1, \dots, T_n, T_1^*, \dots, T_n^*\}$ are $\{0\}$ and \mathfrak{H} . Let S be the (finite dimensional) linear span of $\{I, T_1, \dots, T_n\}$ and let \mathfrak{B} be the C^* -algebra generated by S . Then the identity representation of \mathfrak{B} is irreducible and of course has trivial kernel, so the hypotheses of the implementation theorem will be satisfied when the identity representation is a boundary representation for S . Unfortunately this need not be so in general, even in the “nice” situation where \mathfrak{B} is a type I C^* -algebra (in the case at hand this would imply that \mathfrak{B} contains the compact operators), see 3.5.4 of [1] for an example. Therefore it is of interest to know what additional conditions on $\{T_1, \dots, T_n\}$ will lead to the desired conclusion. The following theorem implies that it is enough to know that some T_i has compact imaginary part.

Theorem 1. *Let S be an irreducible set of operators on \mathfrak{H} such that the norm-closed linear span of $S \cup S^*$ contains a nonzero compact operator. Let φ be a completely positive linear map of the C^* -algebra generated by S into $\mathcal{L}(\mathfrak{H})$ such that $\|\varphi\| \leq 1$ and $\varphi(T) = T$ for all T in S . Then φ is the identity map.*

The earlier version of this theorem [2] required more, namely that every operator in S be compact. This somewhat more general form was obtained by C. A. Akemann and the author. This result, together with the Implementation