

Linear Transformations and Group Representations

J. L. B. COOPER

The fact that the classical integral transforms owe their importance to properties of invariance in certain groups suggest that one investigate what linear transformations have such properties. The group invariances in question take the following form. Let a linear transformation T take a space $A(X)$ of functions on a set X to a space $B(U)$ of functions on U . Let there be a group of transformations $W(g)$ realized by a shift of values on $A(X)$, and a similar group $W^*(g)$ on $B(U)$, that is, $W(g)f(x) = Q(x, g)f(V(g)x)$, where $V(g)$ is a representation of the group by maps of X to itself, $V(g)$ being continuous in g and x : and let $TW(g) = W^*(g)T$.

If the group W is a representation of the additive group of the real line acting on spaces of functions defined on subsets of R it can be shown (1) that under quite general assumptions about the topologies of A and B the possible equations can be reduced to four canonical forms:

- I. $[Tf(x + \alpha)](u) = Tf(x)(u + \alpha), \quad x, u \in R$
- II. $[Tf(x + \alpha)](u) = e^{\alpha\psi(u)}[Tf(x)](u), \quad x \in R, \quad u \in U,$
- III. $[Te^{\alpha\psi(x)}f(x)](u) = Tf(x)(u + \alpha), \quad x \in X, \quad u \in R,$
- IV. $[T(e^{\alpha\psi(x)}f(x))](u) = e^{\alpha\psi(u)}[Tf(x)](u), \quad x \in X, \quad u \in U,$

where X, U are arbitrary subsets of R , and α runs through R .

The nature of the solutions depends somewhat on the spaces concerned. Under quite general conditions, *e.g.* for all L^p spaces, the solutions of I are generalizations of convolutions transforms, and II and III exponential transforms. Those of IV are transpositions of function values.

Some questions here remain unresolved. If instead of requiring that $V(\alpha)x$ be continuous in α we make the requirement that the $W(\alpha)$ be continuous, then other possibilities can occur, for instance in the L^p spaces: and one may ask what $V(\alpha)$ give rise to continuous $W(\alpha)$. Also, there are classical transforms which obey equations of the form II, *e.g.* the inverse Laplace transform, and are not representable by integral transforms. For what spaces and for what transforms is this true?