

A Class of Toeplitz Operators

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This is an interim report on an investigation now in progress into the invertibility of certain Toeplitz operators. Let L^2 and L^∞ denote the Lebesgue spaces of square integrable and essentially bounded functions with respect to Lebesgue measure on the real line. Let H^2 and H^∞ denote the corresponding Hardy spaces for the upper half-plane, interpreted in the usual way as subspaces of L^2 and L^∞ . For φ in L^∞ , the *Toeplitz operator induced by φ* is the operator T_φ on H^2 defined by $T_\varphi f = P(\varphi f)$, where P is the orthogonal projection of L^2 onto H^2 .

We introduce the following classes of complex valued functions on the real line:

C —the continuous functions on \mathbf{R} that extend continuously to the one-point compactification of \mathbf{R} ;

AP —the Bohr almost periodic functions on \mathbf{R} ;

PC —the functions on \mathbf{R} that have right and left limits at all points, including ∞ ;

A_1 —the uniformly closed subalgebra of L^∞ generated by AP and H^∞ .

The spectral properties of T_φ have been successfully analyzed for functions φ in the classes H^∞ [20], C [5], [6], [19], $H^\infty + C$ [7], AP [3], [4], $AP + C$ [9], [10], and PC [5], [11]. In particular, when φ belongs to one of these classes the spectrum of T_φ has a simple description in terms of φ . Our aim in the work under discussion has been to obtain similar results for functions φ in the algebra A_1 . Although we are still very far from our goal, we should nevertheless like to discuss the problem here and to indicate the slight progress we have made so far.

We remark that the algebra A_1 contains all the classes mentioned above except for PC . It also contains all bounded uniformly continuous functions on \mathbf{R} [2, p. 249], and thus it contains the functions in PC that are continuous except at ∞ .

For $a \geq 0$ let ψ_a denote the function $\exp(iax)$. The functions ψ_a belong to H^∞ , and A_1 is the closure in L^∞ of the subalgebra $\cup_{a>0} \overline{\psi_a} H^\infty$. Hence, in attempting to understand T_φ for an arbitrary φ in A_1 , it is natural to restrict one's attention

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