

The “Lifting Theorem” for Intertwining Operators and Some New Applications

BÉLA SZ.-NAGY & CIPRIAN FOIAŞ

1. It is well known (see e.g. [2]) that for every contraction T on a Hilbert space H there exists an isometry V on some Hilbert space K containing H as a subspace, such that

$$(1) \quad TP = PV \text{ (on } K), \text{ and } K = H \vee VH \vee V^2H \vee \dots,$$

P denoting the orthogonal projection from K onto H . These properties determine K and V up to an isometric isomorphism: V is called the minimal isometric dilation of T .

The “Lifting Theorem” for intertwining operators reads as follows:

Theorem. *Let $T_i (i = 1, 2)$ be contractions on H_i , with the respective minimal isometric dilations V_i on K_i . Let X be an operator $H_2 \rightarrow H_1$ such that*

$$(2) \quad T_1X = XT_2.$$

Then there exists an operator $Y: K_2 \rightarrow K_1$ such that

$$(3) \quad V_1Y = YV_2, \quad P_1Y(I - P_2) = 0,$$

and

$$(4) \quad P_1Y|_{H_2} = X.$$

Moreover, we can require that $\|Y\| = \|X\|$. Conversely, every operator $Y: K_2 \rightarrow K_1$ satisfying (3) gives rise, by (4), to an operator X satisfying (2).

(The “converse” part is obvious.)

This theorem was found and published early in 1968 by Sz.-Nagy and Foiaş [4].

The proof in [4] (see also [2]) begins by multiplying equation (2) from the right by P_2 ; using (1) we obtain

$$(2') \quad T_1X' = X'V_2, \text{ where } X' = XP_2: K_2 \rightarrow H_1.$$

In this equation V_2 is already an isometry and all we have to do is to find an operator $Y: K_2 \rightarrow K_1$ such that

$$V_1Y = YV_2, \quad P_1Y = X', \quad \|Y\| = \|X'\|;$$