

Vitushkin's Localization Operator

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An obvious difficulty with function theory is that we cannot use C^∞ multipliers or partitions of unity. This difficulty can be avoided using Vitushkin's localization operator, which employs a trick frequently used in several complex variables. In this expository talk I will discuss this operator and give a typical application.

Let φ be C^∞ with compact support X , and let $f \in L^\infty(\mathbb{C})$. Write

$$(T_\varphi f)(z) = f(z)\varphi(z) + \frac{1}{\pi} \iint \frac{f(\zeta)}{\zeta - z} \frac{\partial \varphi}{\partial \bar{\zeta}} d\zeta d\bar{\zeta},$$

$$\zeta = \xi + i\eta, \quad \frac{\partial}{\partial \bar{\zeta}} = \frac{1}{\partial} \left(\frac{\partial}{\partial \xi} + i \frac{\partial}{\partial \eta} \right).$$

To clarify that T_φ does, we use the notation of distribution theory. By Green's Theorem $T_\varphi f = \varphi f - u$, where $u \in L^\infty$, $u(\infty) = 0$ and

$$\frac{\partial u}{\partial \bar{z}} = f \frac{\partial \varphi}{\partial \bar{z}} \quad (\text{weakly}).$$

Thus

$$\frac{\partial}{\partial \bar{z}} T_\varphi f = \varphi \frac{\partial f}{\partial \bar{z}} \quad (\text{weakly}).$$

Weyl's Lemma then implies

(i) $T_\varphi f$ is analytic off X , and wherever f is analytic.

(ii) $f - T_\varphi f$ is analytic on the interior of $\{\varphi = 1\}$.

Of course these are easy to verify directly.

Now let D be open, $\lambda \in \partial D$ and U a neighborhood of λ . Suppose $f \in H^\infty(D \cap U)$, the ring of bounded analytic functions on $D \cap U$, and declare $f = 0$ elsewhere. Center at λ two discs Δ and $\tilde{\Delta}$ of radii δ and 2δ , with $\tilde{\Delta} \subset U$. Let $\varphi \equiv 1$ on Δ , $\varphi \equiv 0$ off $\tilde{\Delta}$. Taking $F = T_\varphi f$ plus a constant we have the

Extension Lemma. *There is $F \in H^\infty(D)$ such that $F - f$ is continuous and zero at λ .*